Learning and Price Discovery in a Search Model*

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Abstract

We develop a dynamic matching and bargaining game with aggregate uncertainty about the relative scarcity of a commodity. We use our model to study price discovery in a decentralized exchange economy: Traders gradually learn about the state of the market through a sequence of multilateral bargaining rounds. We characterize the resulting equilibrium trading patterns. We show that equilibrium outcomes are approximately competitive when frictions are small. Therefore, prices aggregate information about the scarcity of the traded commodity; that is, prices correctly reflect the commodity's economic value.

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1 Introduction

General equilibrium theory famously states that—under certain assumptions—there exists a vector of prices such that all markets clear. It fails, however, to explain how the market-clearing ("Walrasian") price vector comes about. The literature on dynamic matching and bargaining games, pioneered by Rubinstein and Wolinsky (1985) and by Gale (1987), aims to fill this gap in the foundations for general equilibrium theory. It addresses the question of how prices are formed in decentralized markets and whether these prices are Walrasian. Existing models of dynamic matching and bargaining, however, assume that market demand and supply—and, hence, the market-clearing price—are common knowledge among traders. This assumption is restrictive because markets have been advocated over central planning precisely on the grounds of their supposed ability to "discover" the equilibrium prices by eliciting and aggregating information that is dispersed in the economy; see Hayek (1945). By construction, existing models that take market-clearing prices to be common knowledge remain silent about whether this argument is correct and whether markets can indeed solve the price discovery problem.

We develop a dynamic matching and bargaining game to study price discovery in a decentralized market. We relax the standard assumption that the aggregate state of the market is common knowledge. In our model, individual traders are uncertain about market demand and supply. No individual trader knows the relative scarcity of the good being traded. We analyze the resulting patterns of trade and learning that emerge in equilibrium. We ask whether traders eventually learn the relevant aggregate characteristics and whether prices accurately reflect relative scarcity when frictions are small.

Our model is set in discrete time. In every period, a continuum of buyers and sellers arrives at the market. All buyers are randomly matched to the sellers, resulting in a random number of buyers that are matched with each seller. Each seller conducts a second-price sealed-bid auction. At the end of each round, successful buyers and sellers leave the economy. Unsuccessful traders leave the market with some exogenous exit probability; otherwise they remain in the market to be rematched in the next period. The exogenous exit rate makes waiting costly and is interpreted as the "friction" of trade.

The defining feature of our model is uncertainty about a binary state of nature. The realized state is unknown to the traders and does not change over time. For each state of nature, we consider the corresponding steady state of the market. The state of nature determines the relative scarcity of the good. Depending on the state of nature, the mass of incoming buyers is either large or small, whereas the mass of incoming sellers is independent of the state of nature. The larger the mass of entering buyers is relative to the mass of entering sellers, the scarcer is the good. Every buyer receives a noisy signal upon birth. Moreover, after every auction, the losing buyers obtain additional information regarding the state, because they are able to draw an inference from the fact that their respective bids lost; losing bidders do not observe other buyers' bids.

Formally, our model combines elements from Satterthwaite and Shneyerov (2008) and Wolinsky (1990). Specifically, the matching technology and the bargaining protocol are adapted from Satterthwaite and Shneyerov, and we consider an unknown binary state of nature that does not change over time and study the resulting steady-state outcome for each realization as in Wolinsky (1990). In our base model, buyers and sellers are homogenous and auctions are without reserve price, so that sellers take no actions. We relax these assumptions later when we consider an extension with heterogeneous buyers and allow for reserve prices. In the following, we first describe our findings for the base model and then return to the extension at the end of the Introduction.

Our base model allows a particularly instructive analysis, because we can characterize equilibrium learning and bargaining strategies explicitly. The buyers shade their bids to account for the opportunity cost of foregone continuation payoffs. Moreover, despite the fact that the consumption value of the good is known, the fact that continuation payoffs depend on the unknown common state of nature makes the buyers' preferences interdependent and introduces an endogenous common value element. A resulting winner's curse leads to further bid shading: Winning an auction implies that on average fewer bidders are participating and that the participating bidders are more optimistic about their continuation payoff. Both of these facts imply a lower value of winning the good than expected prior to winning. Countervailing the winner's curse is a "loser's curse." The role of the loser's curse for information aggregation in large double auctions was identified by Pesendorfer and Swinkels (1997). In our model, losing an auction implies that on average more bidders are participating and that the participating bidders are more pessimistic about their continuation payoff. The loser's curse implies that bidders become more pessimistic and raise their bids after repeated losses over time.[1]

We are particularly interested in the characterization of the equilibrium when the exogenous exit rate is small, which is interpreted as the frictionless limit of the decentralized market. Our main result shows that the limit outcome approximates the Walrasian outcome relative to the realized aggregate state of the market. If the realized state is such that the mass of incoming buyers exceeds the mass of incoming

[1] Thus, the loser's curse refers to the effect of the learning dynamics *over time*, whereas the winner's curse refers to bid shading *in each period*.

sellers, the resulting limit price at which trade takes place is equal to the buyers' willingness to pay; if the realized state is such that the mass of incoming buyers is smaller than the mass of incoming sellers, the price is equal to the seller's costs. Therefore, prices aggregate information about the scarcity of the traded commodity; that is, prices reflect the commodity's economic value.

Our analysis reveals how the winner's curse and the loser's curse shape equilibrium outcomes and information aggregation. We show that as the exit rate vanishes, entrants' initial bids are dominated by the winner's curse and the buyers bid for an increasing number of periods as if they are certain that the continuation value is maximal. Eventually, however, the loser's curse becomes strong enough so that those buyers who have lost in a sufficiently large number of periods raise their bids over time. Specifically, the number of periods in which buyers bid the low price diverges to infinity. However, the number of periods in which buyers bid low grows more slowly than the rate at which the exit rate goes to zero. Therefore, bids become high sufficiently fast relative to the exit rate. This ascending bid pattern ensures that actual transaction prices are equal to the sellers' costs if the realized state is such that the buyers are on the short side of the market and actual transaction prices are equal to the buyers' valuations if the buyers are on the long side of the market.

We compare the trading outcome of our model to the trading outcome that would result if the state were known. This comparison allows us to isolate the effect of uncertainty on the trading outcome. Intuitively, uncertainty pushes prices away from their competitive level through two mechanisms. First, buyers underestimate their continuation payoff in the low state (and, hence, overbid relative to complete information) and buyers overestimate their continuation payoff in the high state (and, hence, underbid relative to complete information). Second, the winner's curse depresses buyer's bids relative to complete information. Thus, it should not be surprising that the prices are more distorted when there is aggregate uncertainty than when there is not. Formally, we study the relative magnitude of the distortions of the expected trading prices when frictions become small, by deriving the relative rates of convergence with and without uncertainty. We find that, in the high state, the presence of uncertainty slows down convergence considerably, whereas convergence is at the same rate in the low state. Our finding indicates that competitive equilibrium may be a (much) better approximation of markets in which demand and supply conditions are well known than when they are not. This finding, together with our observations about the implications of the winner's curse and the loser's curse, demonstrate that uncertainty has a signifcant effect on the bidding behavior and the trading outcome.

In an extension of our base model, we introduce heterogeneous valuations among buyers and allow the seller to set a reserve price. The reserve price is observed by the buyers the seller is matched with and becomes a signal of the seller's private belief about the state of nature. This signaling possibility introduces a potential multiplicity problem. To deal with this problem, we assume that buyers' beliefs are passive following the observation of off-equilibrium reserve prices. We characterize equilibria when the exit rate is small. We show that the equilibrium trading outcome is competitive in each state. Our proof utilizes arguments developed in Lauermann (2012). These arguments are significantly extended to account for the dual problem of uncertainty and signaling. The extension demonstrates that the base model's simplifications are not critical to the results. Moreover, with strategic agents on both sides of the market, the market may turn into a two-sided war of attrition, with buyers insisting on low prices (bidding low for many periods) and sellers insisting on high prices (setting high reserve prices). This possibility is absent in the base model.

In the following section, we discuss our contribution to the literature. In Section 3 we introduce the base model. We provide existence and uniqueness results for steady-state equilibria in monotone bidding strategies in Section 4. We also provide some preliminary characterization of equilibrium. Proving the existence of equilibrium is a non-trivial problem in a search model with aggregate uncertainty because of the endogeneity of the distribution of population characteristics (beliefs in our model); see Smith (2011). We show that, in our model, the steady-state distribution of beliefs can be constructed using an intuitive recursive algorithm. Some techniques that we develop in this paper might be useful more generally.[2] Our main result on convergence to the competitive outcome is stated in Section 5. We discuss our extension to an economy with heterogeneous buyers and strategic sellers in Section 6. Section 7 provides a discussion of extensions and conclusion. The Appendix contains the proofs and especially the analysis of the model with heterogeneous buyers and strategic sellers. A supplementary online Appendix collects some technical results about the steady-state stock and the proof of existence of equilibrium for the base model.[3]

2 Contribution to the Literature

We contribute to a body of research that studies the foundations for general equilibrium through the analysis of dynamic matching and bargaining games, which was

[2]For example, we demonstrate a failure of the monotone likelihood ratio property to aggregate in auctions with a random number of bidders; see Section 1.3 of the online Appendix.

[3]At http://sites.google.com/site/slauerma/Price-Discovery-Online-Appendix.pdf

initiated by Rubinstein and Wolinsky (1985) and Gale (1987).[4] A central question is whether a fully specified "decentralized" trading institution leads to outcomes that are competitive when frictions of trade are small. Well-known negative results by Diamond (1971) and Rubinstein and Wolinsky (1985) have demonstrated that this question is not trivial. Studying foundations is important for *positive* theory in order to understand under which conditions markets can and cannot be well approximated by competitive analysis, and for *normative* theory in order to understand what trading institutions are able to decentralize desirable allocations.[5]

In existing models of dynamic matching and bargaining, market demand and supply are known. Thus, each market participant can individually compute the market-clearing price before trading. In the early matching and bargaining literature, the preferences and endowments of each individual trader were typically assumed to be observable. Satterthwaite and Shneyerov (2007, 2008) introduce a model with private information. Because they assume a continuum of agents, the realized distribution of preferences is known in their model, so there is idiosyncratic but no aggregate uncertainty.

The absence of aggregate uncertainty from existing dynamic matching and bargaining games is a substantial restriction and considering uncertainty is important for at least two reasons. First, assuming that aggregate market conditions are known to all participants is unrealistic in many markets. Our model allows us to study those markets in which this assumption is not met. Second, as argued before, price discovery has been highlighted as an integral function of markets. Our model allows us to investigate whether and under which conditions decentralized markets can indeed serve this function.

Modeling uncertainty about market demand and supply leads to novel conceptual challenges. First, we need to characterize the endogenous distribution of beliefs in the economy. This is a key difficulty that has blocked progress on equilibrium learning in search models so far. Second, when buyers and sellers bargain over the price their common outside option is to continue searching for other partners. This introduces an endogenous common value element into the bargaining game. Bargaining with common values is known to cause difficulties because of a multiplicity-of-equilibrium problem. In our base model, we propose a combination of assumptions that minimizes the impact of these problems. The extension to heterogenous buyers and strategic sellers takes these challenges head-on.

A recent contribution by Majumdar, Shneyerov and Xie (2011) is the only other

[4] For recent contributions, see for example Satterthwaite and Shneyerov (2008), Shneyerov and Wong (2010), and Kunimoto and Serrano (2004) and the references therein.

[5] For an excellent discussion of strategic foundations for general equilibrium, see the introductory chapter in Gale (2000).

paper that considers a dynamic matching and bargaining game in which market demand and supply depend on an unknown state of nature. Their key assumption is that traders from each market side are subjectively certain that the state of nature is in their favor. This assumption allows them to prove existence and provide a full characterization of equilibrium even when traders are heterogenous. The assumption is restrictive because it implies that there is in fact no uncertainty from the viewpoint of each trader.

Our paper is also related to work on matching and bargaining with exogenously assumed *common values*. In these models, preferences depend on an unknown state, and, consequently, these models are used to study foundations for Rational Expectations Equilibria. By contrast, we study the foundations for competitive equilibrium in a standard exchange economy.

Particularly prominent contributions to search with common values are Wolinsky (1990) and later work by Blouin and Serrano (2001).[6] These contributions provide negative convergence results and uncover a fundamental problem of information aggregation through search: As frictions vanish, traders can search and experiment at lower costs. This might seem to make information aggregation simple. However, it also implies that traders increasingly insist on favorable terms—the buyers on low prices and the sellers on high prices—turning the search market into "a vast war of attrition" (Blouin and Serrano (2001, p. 324)). This insistence on extreme positions makes information aggregation difficult even when search frictions are small. In our model, the winner's curse implies a similar effect: When the exit rate vanishes, the buyers bid low and insist on a price equal to the sellers' cost for an increasingly large number of periods. Yet, in our setting, this "insistence problem" is overcome by the opposing loser's curse, as discussed before.

Wolinsky (1990) and Blouin and Serrano (2001) assume that traders can choose only between two price offers (bargaining postures). This assumption makes their models tractable. It is an open question whether trading outcomes in these models are competitive if the frictions are small and if the restriction on the set of prices is not imposed. Golosov, Lorenzoni and Tsyvinski (2011) consider a related model of search with common values in which the traded good is divisible. They do not impose a restriction on the set of price offers. The friction in their model is an exogenous probability that trading stops in any given period. They show that equilibrium outcomes approximate ex-post efficient outcomes in the event that the game has not stopped for a sufficiently large number of periods. Golosov et al. study the

[6] Serrano and Yosha (1993) consider a related problem with one-sided private information and Gottardi and Serrano (2005) consider a "hybrid" model of decentralized and centralized trading. Lauermann and Wolinsky (2011) study information aggregation if a single, privately informed buyer searches among many sellers.

trading outcome with a fixed, positive stopping probability and they do not study the question whether outcomes become competitive in the "frictionless" limit when the stopping probability is small.[7]

There is a large body of related work on the foundation for rational expectation equilibrium in centralized institutions in which all traders simultaneously interact directly (see, e.g., the work on large double auctions by Reny and Perry (2006) and Pesendorfer and Swinkels (1997, 2000)) and on the behavior of traders in financial markets (e.g., Kyle (1989), Ostrovsky (2011) and Rostek and Weretka (2012)). The assumption of a central price formation mechanism distinguishes this literature from dynamic matching and bargaining games in which prices are determined in a decentralized manner through bargaining.

Finally, our work is related to the literature on social learning (Banerjee and Fudenberg (2004)), the recent work on information percolation in networks (Golub and Jackson (2010)), and information percolation with random matching (Duffie and Manso (2007)). In the latter model, agents who are matched observe each other's information. In our model, the amount of information that one bidder learns from other traders is endogenous and depends on the action (bid) that they choose.

3 The Base Model and Equilibrium

3.1 Setup

There are a continuum of buyers and a continuum of sellers present in the market. In periods $t \in \{..., -1, 0, 1, ...\}$, these traders exchange an indivisible, homogeneous good. Each buyer demands one unit, and the buyers have a common valuation v for the good. Each seller has one unit to trade. The common cost of selling is $c = 0$. Trading at price p yields payoffs $v - p$ and $p - c$, respectively. The valuation exceeds the cost, so there are gains from trade. Buyers and sellers maximize expected payoffs.

Similar to Wolinsky (1990), there are two states of nature, a high state and a low state $w \in \{H, L\}$. Both states are equally likely. The realized state of nature is fixed throughout and unknown to the traders. For each realization of the state of nature, we consider the corresponding steady-state outcome, indexed by w. The state of nature determines the constant and exogenous number of new traders who enter the market (the *flow*), and, indirectly, it also determines the constant and endogenous number of traders in the market (the *stock*). In the low state, the mass of buyers entering each period is d^L, and, in the high state, it is d^H. More buyers

[7] As explained in Golosov et al., ex-post efficiency of the outcome in the event that the game does not stop does not imply that this is the rational expectations equilibrium relative to the initial allocation.

enter in the high state, $d^H > d^L$. The mass of sellers who enter each period is the same in both states and is equal to one. We are most interested in the case where $d^H > 1 > d^L$, so that buyers are on the long side of the market in high state and on the short side of the market in the low state.

The buyers are characterized by their beliefs $\theta \in [0,1]$, the probability that they assign to the high state. In the following, we often refer to θ as the *type* of a buyer. Each buyer who enters the market privately observes a noisy signal and forms a posterior based on Bayesian updating. In state w, the posteriors of the entering buyers are assumed to be distributed on the support $[\underline{\theta}, \overline{\theta}]$, with cumulative distribution functions G^H and G^L, respectively. The distributions are continuous and admit continuous probability density functions, g^H and g^L. Notice that using Bayes' rule the distributions must be such that $\theta = \frac{d^H g^H(\theta)}{d^H g^H(\theta) + d^L g^L(\theta)}$, or, equivalently, the likelihood ratio satisfies

$$\frac{\theta}{1-\theta} = \frac{d^H}{d^L} \frac{g^H(\theta)}{g^L(\theta)}.$$

For a buyer, the mere fact of entering the market contains news because the inflow is larger in the high state. Conditional on entering the market, a buyer is pessimistic and believes that the high state is more likely than the low state. This is expressed by the likelihood ratio $d^H/d^L > 1$.[8]

We assume that the lower and upper bounds of the support $[\underline{\theta}, \overline{\theta}]$ are such that $1/2 \leq \underline{\theta} < \overline{\theta} < 1$. The assumption that $1/2 \leq \underline{\theta}$ is needed to ensure monotonicity of a certain posterior; see the remarks following Lemma 3. Substantively, this assumption is consistent with signals being sufficiently noisy, so that even the most favorable signal $\underline{\theta}$ is not strong enough to overturn the initial pessimism of an entering buyer.[9] The assumption $\bar{\theta} < 1$ implies that signals are boundedly informative.[10] Each period unfolds as follows:

1. Entry occurs (the *"inflow"*): A mass one of sellers and a mass d^w of buyers enter the market. The buyers privately observe signals, as described before.

2. Each buyer in the market (the *"stock"*) is randomly matched with one seller. A seller is matched with a random number of buyers. The probability that a

[8] To formally define updating based on entering the market, suppose that there is a *potential* set of buyers of mass d, $d \geq d^H$. In state w, a mass d^w of the potential buyers actually enters the market. Alternatively, one can simply interpret d^H/d^L as the prior of an entering buyer. For games with population uncertainty and updating about an unkown state of nature, see Myerson (1998) and, especially, Milchtaich (2004).

[9] When signals are very uninformative, most buyers' beliefs are close to their prior conditional on being born, which is $d^H/(d^H + d^L) > 1/2$.

[10] We endow buyers with initial signals to ensure existence of a pure strategy equilibrium. The interval $[\underline{\theta}, \bar{\theta}]$ can be arbitrarily small. Our results also hold if the initial signal is fully uninformative. In this case, bidders will mix over bids in their first auction.

seller is matched with $k = 0, 1, 2, ...$ buyers is Poisson distributed[11] and equal to $e^{-\mu}\mu^k/k!$, where μ is the endogenous ratio of the mass of buyers to the mass of sellers in the stock. We sometimes refer to μ as a measure of market "tightness." The expected number of buyers who are matched with each seller is equal to μ, of course.

3. Each seller runs a sealed-bid second-price auction with no reserve price. The buyers do not observe how many other buyers are matched with the same seller. The bids are not revealed ex post, so bidders learn only whether they have won with their submitted bid.

4. A seller leaves the market if its good is sold; otherwise, the seller stays in the stock with probability $\delta \in [0, 1)$ to offer its good in the next period. A winning buyer pays the second highest bid, obtains the good, and leaves the market. A losing buyer stays in the stock with probability δ and is matched with another seller in the next period. Those who do not stay exit the market permanently. A trader who exits the market without trading has a payoff of zero.

5. Upon losing, the remaining buyers update their beliefs based on the information gained from losing with their submitted bids. The remaining buyers and sellers who neither traded nor exited stay in the market. Together with the inflow, these traders make up the stock for the next period.

On the individual level, the exit rate $1 - \delta$ acts similar to a discount rate: Not trading today creates a risk of losing trading opportunities with probability $1-\delta$. On the aggregate level, the exit rate ensures that a steady state exists for all strategy profiles; see Nöldeke and Tröger (2009). Traders do not discount future payoffs beyond the implicit discounting of the exit rate. The matching technology and the bargaining protocol is adapted from Satterthwaite and Shneyerov (2008). The main difference is that they assume that buyers and sellers are heterogeneous and that sellers can set an ex-post reserve price. In Section 6, we introduce buyer heterogeneity and we allow sellers to set an ex-ante (and observable to the buyers) reserve price.

3.2 Steady-State Equilibrium

We study steady-state equilibria in stationary strategies so that the distribution of bids depends only on the state and not on (calendar) time. An immediate conse-

[11] This distribution is consistent with the idea that there are a large number of buyers who are independently matched with sellers. The resulting distribution of the number of buyers matched with a seller is binomial. When the number of buyers and sellers is large, the binomial distribution is approximated by the Poisson distribution.

quence is that in any period the set of optimal bids of a buyer depends only on the current belief about the likelihood of being in the high state.

We restrict attention to symmetric and pure strategy equilibria where the bid is a strictly increasing function of the belief of the buyer and where the distribution of beliefs is sufficiently "smooth," as defined below. A symmetric steady-state equilibrium is a vector $(\Gamma^H, \Gamma^L, S^H, D^H, S^L, D^L, \beta, \theta^+)$. Next, we describe each of these components. First, the distributions of beliefs are given by atomless cumulative distribution functions Γ^w. We assume that each function Γ^w is absolutely continuous and nondecreasing.[12] Furthermore, we assume that Γ^w is piecewise twice continuously differentiable.[13] These assumptions ensure that we can choose a density, denoted γ^w, that is right continuous on $[0, 1)$.[14]

The masses of buyers and sellers in the stock are D^w and S^w. The bidding strategy β is a strictly increasing function and maps beliefs from $[0, 1]$ to nonnegative bids. We often use the generalized inverse of β, given by $\beta^{-1}(b) = \inf\{\theta | \beta(\theta) \geq b\}$, where $\beta^{-1}(b) = 1$ if $\beta(\theta) < b$ for all θ. Finally, $\theta^+(x, \theta)$ is the posterior of a buyer with initial belief θ conditional on losing against buyers with beliefs above x.

We characterize the equilibrium requirements for these objects; a formal definition of equilibrium follows at the end of this section. Let $\theta_{(1)}$ denote the first-order statistic of beliefs in any given match. We set $\theta_{(1)} = 0$ if there is no bidder present. $\Gamma^w_{(1)}$ denotes the c.d.f. of the first-order statistic in state w; that is, $\Gamma^w_{(1)}(x)$ is the probability that the highest belief in the auction is below x. The event in which all the buyers have a belief below x includes the event in which there are no buyers present at all. The probability of this event is $\Gamma^w_{(1)}(0)$ by our assumption that there is no atom in the distribution of beliefs at zero. The Poisson distribution implies $\Gamma^w_{(1)}(0) = e^{-\mu^w}$, where $\mu^w = D^w/S^w$ as defined before. The fact that this probability is positive implies that the buyers must have positive expected payoffs because any buyer has some probability of being the sole bidder and receiving the good at a price of zero. In general, the first-order statistic of the distribution of beliefs is given by

$$\Gamma^w_{(1)}(\theta) = e^{-\mu^w(1-\Gamma^w(\theta))}. \tag{1}$$

Intuitively, $\mu^w(1 - \Gamma^w(\theta))$ is the ratio of the mass of buyers having belief above θ

[12] Note that this is a restriction on the kind of steady-state equilibria that we study.

[13] A function is piecewise twice continuously differentiable on $[0, 1]$ if there is a partition of $[0, 1]$ into a countable collection of open intervals and points such that the function is twice continuously differentiable on each open interval. Moreover, we require that the set of non-differentiable points has no accumulation point except at one. Smoothness ensures that we can work conveniently with densities.

[14] We believe that these restrictions are without loss of generality. The restriction to symmetric and pure strategies is without loss of generality by the uniqueness of the optimal bids, whenever belief distributions are atomless and bidding strategies are strictly increasing. However, we have not been able to show that all equilibria have these latter properties.

to the mass of sellers, and $e^{-\mu^w(1-\Gamma^w(\theta))}$ is the probability that the seller is matched with no buyer having such belief. Let $\Gamma^{\theta}_{(1)}(x) = \theta\Gamma^{H}_{(1)}(x) + (1-\theta)\Gamma^{L}_{(1)}(x)$ be the unconditional probability that the highest belief is below x if the probability of the high state is θ.

We derive the posterior upon losing. Given the assumption that bidding strategies are strictly increasing, losing with a bid b implies that there was some bidder in the match with a belief above $x = \beta^{-1}(b)$. Bayes' rule for the posterior θ^+ requires that

$$\theta^+(x,\theta) = \frac{\theta(1-\Gamma^{H}_{(1)}(x))}{1-\Gamma^{\theta}_{(1)}(x)} \tag{2}$$

if $1-\Gamma^{\theta}_{(1)}(x) > 0$. Otherwise, we set $\theta^+(x,\theta) \equiv \sup\{(\theta^+(x',\theta)) | \, x' : 1-\Gamma^{\theta}_{(1)}(x') > 0\}$, which is well defined by monotonicity of $\Gamma^{w}_{(1)}$. This particular choice of the "off-equilibrium" belief does not affect our analysis.[15]

To derive the steady-state conditions for the stock, suppose that the mass of sellers is S^w today. A seller trades if and only if matched with at least one buyer. Tomorrow's population of sellers therefore consists of the union of those sellers who were not matched with any buyer and the newly entering sellers. In steady state, these two populations must be identical, requiring

$$S^w = 1 + \delta\Gamma^{w}_{(1)}(0)\, S^w. \tag{3}$$

The inflow of buyers having type less than θ is $d^w G^w(\theta)$. The stationarity condition is

$$D^w\Gamma^w(\theta) = d^w G^w(\theta) + \delta D^w \int_{\{\tau:\theta^+(\tau,\tau)\le\theta\}} \left(1-\Gamma^{w}_{(1)}(\tau)\right) d\Gamma^w(\tau). \tag{4}$$

The steady-state mass of buyers in the stock having a type below θ is equal to $D^w\Gamma^w(\theta)$. This mass has to be equal to the mass of the buyers in the inflow with type less than θ (the first term on the right-hand side) plus the mass of buyers who lose, survive, and update to some type less than θ (the second term).[16]

Let $V(\theta)$ denote the value function, which is equal to

$$\max_b v\Gamma^{\theta}_{(1)}(0) + \int_{0^+}^{\beta^{-1}(b)} (v-\beta(\tau))\, d\Gamma^{\theta}_{(1)}(\tau) + \delta(1-\Gamma^{\theta}_{(1)}(\beta^{-1}(b)))V(\theta^+), \tag{5}$$

[15] When restricting beliefs to the two states of nature, we implicitly assume that following an off-equilibrium event—the only such event is losing with a bid above the highest equilibrium bid—a buyer continues to believe that all other buyers play according to their equilibrium strategies.

[16] For the purpose of this paper, the steady-state model is *defined* by (3) and (4). Formally, these equations are taken as the primitives of our analysis and they are not derived from some stochastic matching process. This allows us to avoid well-known measure theoretic problems with a continuum of random variables. These problems can be solved, however, at the cost of additional complexity; see Duffie and Sun (2007).

where $\theta^{+} = \theta^{+}\left(\beta^{-1}(b), \theta\right)$. A bidding strategy β is optimal if $b = \beta(\theta)$ solves the maximization problem (5) for every θ.

A steady-state equilibrium in symmetric, strictly increasing bidding strategies with an atomless distribution of types (an *equilibrium* from now on) consists of (i) masses of buyers and sellers, S^H, D^H, S^L, D^L, and distribution functions Γ^H, Γ^L such that the steady-state conditions (3) and (4) hold for all θ; (ii) an updating function θ^{+} that is consistent with Bayes' rule (2); (iii) a strictly increasing bidding function β that is optimal (maximizes (5)).

4 Characterization and Existence of Equilibrium

4.1 The Equilibrium Stock

The following lemmas establish necessary implications of equilibrium for the steady-state stock. Generally, characterizing stocks in equilibrium search models is difficult because of an intricate feedback between stocks and strategies, which requires determining these two objects simultaneously. In our model, however, we can "decouple" the stock from the strategies. This is because the bidding strategy is strictly increasing: The identity of the winning bidder as well as the updated belief is the same for all strictly increasing bidding strategies. We now describe properties of the stock, assuming (and verifying later) that a monotone equilibrium exists. All proofs of the results from this section are in the supplementary online Appendix, with the exception of the proof of the following lemma.

Lemma 1 (Unique Masses.) *For each state w, there are unique masses of buyers D^w and sellers S^w that satisfy the steady-state conditions. The market is tighter in the high state; that is, $\frac{D^H}{S^H} > \frac{D^L}{S^L}$.*

The lemma is intuitive: The larger mass of buyers in the high state implies that more buyers stay in the market because each buyer has a smaller chance to transact. Moreover, each seller has a higher chance to transact in the high state, so the sellers leave the market more quickly, and there are fewer sellers on the market in the high state.

A distribution of beliefs is said to have the no-introspection property if

$$\frac{\theta}{1-\theta} = \frac{D^H}{D^L} \frac{\gamma^H(\theta)}{\gamma^L(\theta)} \tag{6}$$

for all $\theta < 1$ with $\gamma^L(\theta) > 0$. The condition implies that a buyer does not update based merely on observing its own belief ("introspection"). The following lemma follows from the steady-state conditions.

Lemma 2 (No-Introspection.) *If Γ^w is an atomless and piecewise twice continuously differentiable c.d.f. and if Γ^w satisfies the steady-state conditions given the steady-state masses S^w, D^w, then Γ^w and D^w have the no-introspection property.*

The distribution of beliefs satisfies the "monotone likelihood ratio property" (*MLRP*) if $\gamma^H(\theta'')\gamma^L(\theta') \geq \gamma^H(\theta')\gamma^L(\theta'')$ whenever $\theta'' \geq \theta'$. The no-introspection condition implies the MLRP. Intuitively, observing a buyer with a higher belief makes the high state more likely. The no-introspection condition also implies that the distributions Γ^H and Γ^L have identical support, given that there are no atoms at zero or one.

We use the MLRP to characterize updating. Suppose that $0 < \Gamma^L(\theta) < 1$. The MLRP implies that $1 - \Gamma^H(\theta) > 1 - \Gamma^L(\theta)$. By Lemma 1, $\mu^H > \mu^L$. Therefore, $\mu^H\left(1 - \Gamma^H(\theta)\right) > \mu^L\left(1 - \Gamma^L(\theta)\right)$, the expected number of buyers with belief above θ who are matched with a seller is higher in the high state. From the definition of $\Gamma^w_{(1)}$, $1 - \Gamma^H_{(1)}(\theta) > 1 - \Gamma^L_{(1)}(\theta)$, the likelihood of losing is higher in the high state for any θ. Hence, "losing is bad news," and the posterior conditional on losing satisfies $\theta^+(x,\theta) > \theta$ whenever $0 \leq \Gamma^L(x) < 1$; see Definition (2). An implication is that all the buyers in the stock must have beliefs above the most optimistic type in the inflow $\underline{\theta}$: All of those buyers who have just entered hold beliefs above the cutoff $\underline{\theta}$. For all other buyers in the stock who have entered at least one period before, the finding that $\theta^+(x,\theta) > \theta$ for all θ implies that their beliefs are above $\underline{\theta}$ as well. Therefore, all beliefs in the stock are above the cutoff $\underline{\theta}$.[17]

We also need the posterior conditional on being tied when characterizing optimal bidding. Conditional on state w, the density of the first-order statistic is $\gamma^w_{(1)} = \frac{D^w}{S^w}\gamma^w\Gamma^w_{(1)}$. The unconditional density is $\gamma^\theta_{(1)}(x) = \theta\gamma^H_{(1)}(x) + (1-\theta)\gamma^L_{(1)}(x)$. The posterior of type θ after tying with a buyer with belief x at the top spot is

$$\theta^0(x,\theta) = \frac{\theta\gamma^H_{(1)}(x)}{\gamma^\theta_{(1)}(x)} \tag{7}$$

if $\gamma^\theta_{(1)}(x) > 0$.[18] The next lemma establishes that updating is monotone.

Lemma 3 (Monotonicity of Posteriors.) *Suppose that Γ^w is an atomless and piecewise twice continuously differentiable c.d.f., (i) the monotone likelihood ratio property holds, and (ii) $\mu^H \geq \mu^L > 0$. Then, the posterior upon losing, $\theta^+(x,\theta)$, is*

[17] This informal argument is verified in the proof of Lemma 4.

[18] We extend the definition of the posterior to all types: If $\min\left\{\Gamma^L(x), \Gamma^H(x)\right\} < 1$, we set $\theta^0(x,\theta) = \inf\{\left(\theta^0(x',\theta)\right) | x' \geq x \text{ and } \gamma^\theta_{(1)}(x) > 0\}$ and if $\Gamma^L(x) = \Gamma^H(x) = 1$, we set $\theta^0(x,\theta) = \sup\{\left(\theta^0(x',\theta)\right) | x' \leq x \text{ and } \gamma^\theta_{(1)}(x) > 0\}$. Bidders do not observe whether they are tied, and the particular choice of the extension of Bayes' formula does not affect our analysis.

nondecreasing in x. If, in addition, (iii) $\gamma^H(\theta) \geq \gamma^L(\theta)\frac{\mu^L}{\mu^H}$ for all θ, then the posterior upon being tied, $\theta^0(x,\theta)$, is nondecreasing in x on $[0,1]$.

A standard sufficient condition for monotonicity of the posteriors would be that the first-order statistic $\theta_{(1)}$ inherits the monotone likelihood ratio property of the parent distribution of θ; see, for example, Krishna (2009). However, in contrast to standard auction settings, the MLRP is not inherited here because the first-order statistic is taken from a *random* number of random variables.[19]

Given a symmetric equilibrium, the posterior conditional on losing is $\theta^+(\theta,\theta)$. An implication of Lemma 3 is that this posterior is strictly increasing in θ. The same holds for the posterior conditional on tying, $\theta^0(\theta,\theta)$. This follows from the fact that conditions (i)—(iii) hold in equilibrium: Conditions (i) and (ii) follow from Lemma 2 and Lemma 1, respectively. For Condition (iii), note the following: The support of Γ^w is a subset of $[\underline{\theta},1]$; see the previous remark following $\theta^+(x,\theta) > \theta$. Therefore, the no-introspection property from Lemma 2 and $S^H \leq S^L$ from Lemma 1 together imply $\frac{\gamma^H(\theta)}{\gamma^L(\theta)}\frac{\mu^H}{\mu^L} \geq \frac{\gamma^H(\theta)}{\gamma^L(\theta)}\frac{D^H}{D^L} \geq \frac{\underline{\theta}}{1-\underline{\theta}}$. Finally, the assumption that $\underline{\theta} \geq 1/2$ implies $\frac{\underline{\theta}}{1-\underline{\theta}} \geq 1$; that is, (iii) holds.

We show that the distribution of beliefs of buyers in the stock is unique. Together with the previous finding that the mass of buyers and sellers is unique, the lemma implies that there exists a unique steady-state stock.

Lemma 4 (Uniqueness of the Steady-State Distributions.) *There exists a unique absolutely continuous and piecewise twice continuously differentiable distribution Γ^w that satisfies the steady-state conditions.*

We describe the basic idea of the proof and some of the complications here. To illustrate the construction and the uniqueness argument, let us suppose momentarily that we have found some stock Γ^w, D^w, S^w that satisfies the steady-state conditions and suppose further that the interval of initial beliefs $[\underline{\theta},\overline{\theta}]$ is sufficiently small such that upon updating, $\theta^+(\underline{\theta},\underline{\theta}) > \bar{\theta}$. Therefore, the set of beliefs of buyers who have lost once is above the interval of the initial beliefs. Consequently, the mass of buyers with beliefs below any $\theta' \in [\underline{\theta},\overline{\theta}]$ is just the mass of such buyers in the inflow; that is, $D^w\Gamma(\theta') = d^w G^w(\theta')$. Moreover, the mass of buyers with beliefs above θ' is $D^w - d^w G^w(\theta')$. Therefore, the probability that a buyer with belief θ' loses is $e^{-(D^w - d^w G^w(\theta'))/S^w}$ in each state w. This determines the posterior of θ' after losing once, $\theta^+(\theta',\theta')$. Conversely, for any θ'' from the set of beliefs who lost once—$\left[\theta^+(\underline{\theta},\underline{\theta}),\theta^+(\overline{\theta},\overline{\theta})\right]$—we can find the prior $\hat{\theta}$ such that $\theta^+(\hat{\theta},\hat{\theta}) = \theta''$. Taking

[19] We provide a detailed discussion of the failure of the MLRP of the first-order statistic with a random number of bidders in our supplementary online appendix in Section 1.3.

our observations together, the distribution of beliefs of buyers who have lost once is given by

$$D^w\Gamma^w\left(\theta''\right) = d^w + \delta d^w \int_{\underline{\theta}}^{\hat{\theta}} e^{-(D^w - G^w(\tau))/S^w} dG^w\left(\tau\right) \qquad \forall\theta'' \in \left[\theta^+\left(\underline{\theta},\underline{\theta}\right), \theta^+\left(\overline{\theta},\overline{\theta}\right)\right].$$

The above reasoning suggests that we can construct the population of buyers inductively, starting with the distribution of initial beliefs and then proceeding to the distribution of beliefs of buyers who have lost once, twice, ... and so on. Furthermore, the above arguments suggest that the construction would yield a unique candidate for an equilibrium steady-state stock.

The existence proof is based on induction, following the line of reasoning laid out before. There are two difficulties with the argument, however. First, we have assumed that intervals of beliefs of successive generations of buyers do not overlap. This does not need to be the case. To take care of this problem, we use the fact that the losing probabilities of the lowest type $\underline{\theta}$ are determined by the total masses D^w and S^w, which are unique by Lemma 1. This determines the posterior $\theta^+\left(\underline{\theta},\underline{\theta}\right)$ and implies that the set of buyers with beliefs in $\left[\underline{\theta}, \min\left\{\theta^+\left(\underline{\theta},\underline{\theta}\right), \bar{\theta}\right\}\right]$ is given by the inflow. We then apply similar arguments successively. Second, the construction above uses the fact that posteriors after losing are monotone in priors. However, the argument following Lemma 3 for monotonicity of θ^+ presupposes the steady-state conditions to conclude that no-introspection holds. When we prove the existence of a steady-state stock, we need to directly ensure that the conditions of Lemma 3 hold, which is done in the main technical lemma of the proof, Lemma 15.

4.2 Characterization of Bidding and Existence of Equilibrium

We characterize the equilibrium bidding strategy. Let

$$EU(\theta,\beta|w) = v\Gamma^w_{(1)}\left(0\right) + \int_{\underline{\theta}}^{\theta}\left(v - \beta\left(\tau\right)\right)d\Gamma^w_{(1)}\left(\tau\right) + \delta\left(1 - \Gamma^w_{(1)}\left(\theta\right)\right)EU(\theta^+\left(\theta,\theta\right),\beta|w)$$

denote the expected utility of a bidder with belief θ given a symmetric bidding strategy β, conditional on state w. The unconditional expected payoff is $EU(\theta,\beta|\hat{\theta}) = \hat{\theta}EU(\theta,\beta|H) + (1-\hat{\theta})EU(\theta,\beta|L)$. The function $EU(\theta,\beta|\hat{\theta})$ can be interpreted as type $\hat{\theta}$'s expected (off-equilibrium) payoff from bidding like type θ.

We prove that equilibrium bids must be

$$\beta(\theta) = v - \delta EU\left(\theta^+\left(\theta,\theta\right), \beta|\theta^0\left(\theta,\theta\right)\right). \tag{8}$$

An intuition for this bidding strategy is as follows. By standard reasoning about bidding in second-price auctions, the bid must be "truthful" and equal to the expected payoff from winning conditional on being tied. Here, the expected payoff from winning is equal to the valuation v minus the relevant continuation payoff. For the relevant continuation payoff, note that the strategy adopted from tomorrow onwards is the optimal strategy given the updated belief conditional on having lost, $\theta^+(\theta,\theta)$. We need to evaluate the expected value of that strategy using the posterior conditional on being tied (the "pivotal event"). Therefore, the expected continuation payoff is calculated by evaluating the utility derived from the future bidding sequence of a bidder with belief $\theta^+(\theta,\theta)$, given the posterior probability of the high state conditional on being tied, $\theta^0(\theta,\theta)$. Thus, the relevant continuation payoff is $\delta EU\left(\theta^+(\theta,\theta),\beta|\theta^0(\theta,\theta)\right)$, and buyers optimally "shade" their bids by this amount.

We provide some auxiliary observations. First, the value function is convex in beliefs: Optimal bidding is a decision problem under uncertainty, implying a convex value function by standard arguments from information economics. Second, the envelope theorem dictates a simple relation between EU, V, and the derivative V'.

Lemma 5 (Characterizing the Value Function.) *The value function $V(\theta)$ is convex. At all interior differentiable points of the value function, $V'(\theta)=\frac{\partial}{\partial\hat{\theta}}EU(\theta,\beta|\hat{\theta})|_{\hat{\theta}=\theta}$, and*

$$EU(\theta,\beta|\hat{\theta})=V(\theta)+(\hat{\theta}-\theta)V'(\theta). \tag{9}$$

The following Lemma establishes a unique candidate for the equilibrium bidding function for given continuation payoffs. The lemma follows from rewriting the necessary first-order condition for optimal bids; that is, we determine the derivative of the objective function (5) with respect to b and set it equal to zero.

Lemma 6 (Equilibrium Candidate.) *For almost all types in the support of the distribution of beliefs, in equilibrium*

$$\beta(\theta)=v-\delta V\left(\theta^+(\theta,\theta)\right)+\delta V'\left(\theta^+(\theta,\theta)\right)(\theta^+(\theta,\theta)-\theta^0(\theta,\theta)). \tag{10}$$

We can use Lemma 5 to substitute EU for V' and V in equation (10). After the substitution, the expression for the bidding strategy is as claimed in the equation (8).

We have identified a unique candidate for the equilibrium bidding strategy for given continuation payoffs in this section. We have also proven that there exists a unique steady-state stock in Section 4.1. The following proposition shows that there exists an equilibrium. The exogenous parameters—δ, d^H, and d^L—determine the market outcome in an essentially unique way. The proof is in the online Appendix.

Proposition 1 (Existence and Uniqueness of Equilibrium.) *There exists a steady-state equilibrium in strictly increasing strategies. The equilibrium distribution of beliefs and the value function $V(\theta)$ are unique. For almost all types in the support of the distribution of beliefs, the bidding function is $\beta = v - \delta EU\left(\theta^{+}(\theta,\theta), \beta|\theta^{0}(\theta,\theta)\right)$.*

5 Price Discovery with Small Frictions

We state and prove our main result: as the exit rate becomes small, the equilibrium trading outcome becomes competitive in each state. In particular, all trade between buyers and sellers takes place at the "correct," market-clearing prices.

We define *trading outcomes.* For buyers, the trading outcome in state w consists of the equilibrium probability of winning in an auction (instead of being forced to exit) and the expected price paid conditional on winning, denoted $Q^{w}(\theta)$ and $P^{w}(\theta)$, respectively. For a seller, the trading outcome consists of a probability of being able to sell the good and the expected price received, denoted $Q^{w}(S)$ and $P^{w}(S)$. The inflow defines a large quasilinear economy, where the mass of buyers is d^{w} and the mass of sellers is independent of w and equal to one. A trading outcome is said to be a (perfectly) *competitive outcome* (or Walrasian outcome) relative to the economy defined by the inflow if prices and trading probabilities are as follows. If $d^{w} < 1$ (i.e., if buyers are on the short side of the market), then $P^{w}(\theta) = P^{w}(S) = 0$, $Q^{w}(\theta) = 1$, and $Q^{w}(S) = d^{w}$. If $d^{w} > 1$ (i.e., if buyers are on the long side of the market), then $P^{w}(\theta) = P^{w}(S) = v$, $Q^{w}(\theta) = 1/d^{w}$, and $Q^{w}(S) = 1$. We do not characterize the competitive outcome in the case in which both market sides have equal size, $d^{w} = 1$. If an outcome is competitive, it is necessarily an efficient outcome relative to the economy defined by the inflow.

We consider the trading outcome when the exit rate is small. Let $\{\delta_k\}_{k=1}^{\infty}$ be a sequence such that the exit rate converges to zero, $\lim(1-\delta_k) = 0$. Intuitively, a smaller exit rate corresponds to a smaller cost of searching. To interpret our results, it might be helpful to observe that decreasing the exit rate is equivalent to increasing the speed of matching.[20] We know that an equilibrium exists for each δ_k. Pick any such equilibrium and denote the corresponding equilibrium magnitudes by β_k,Γ_k^H, Γ_k^L, D_k^H,P_k^w, Q_k^w, and so on. A sequence of trading outcomes converges to the competitive outcome relative to the economy defined by the inflow in state w if the sequence of outcomes converges pointwise for all θ and for S.

[20] Let Δ_k denote the length between time periods and let d denote the (fixed) exit probability per unit of time. With $1-\delta_k = \Delta_k d$, one can interpret a decrease in the exit rate $1-\delta_k$ as a decrease in Δ_k. In this interpretation, the market friction $1-\delta_k$ arises because it takes time Δ_k to come back to the market after a loss. As this time lag Δ_k goes to zero, the friction vanishes.

Proposition 2 (Price Discovery with Small Frictions.) *For any sequence of vanishing exit rates and for any sequence of corresponding steady-state equilibria in strictly increasing strategies, the sequence of trading outcomes converges to the competitive outcome for each state of nature.*

We illustrate the proposition through a few observations. First, we restate the implications of the proposition in terms of the limit of the value function.

Corollary 1 (Limit Payoffs.) *For any sequence of equilibria for* $\lim_{k\to\infty}(1-\delta_k)=0$*:* $\lim_{k\to\infty} V_k(\theta) \equiv v$ *if* $d^L < d^H < 1$*;* $\lim_{k\to\infty} V_k(\theta) \equiv 0$ *if* $1 < d^L < d^H$*; and* $\lim_{k\to\infty} V_k(\theta) = (1-\theta)\,v + \theta 0$ *if* $d^L < 1 < d^H$*.*

The corollary is immediate and the proof is omitted. Intuitively, the short side of the market captures the surplus from trading. Moreover, the corollary states that the value function is no longer convex but linear in the limit. Information loses its value when the friction of trade is small.

The following result is the main intermediate step towards proving the Proposition. The lemma illustrates some of the main forces at work.

Lemma 7 (Limit Market Population.) *For any sequence of equilibria for* $\lim_{k\to\infty}(1-\delta_k) = 0$ *the following statements hold: (i)*

$$\lim_{k\to\infty}\frac{D_k^w}{S_k^w} = \begin{cases} 0 & \text{if } d^w < 1 \\ \infty & \text{if } d^w > 1. \end{cases}$$

(iia) If $d^w < 1$*, the probability of being the sole bidder becomes one,* $\lim_{k\to\infty} e^{-D_k^w/S_k^w} = 1$*.*

(iib) If $d^w > 1$*, the probability of being the sole bidder converges to zero and it converges to zero faster than the exit probability,* $\lim_{k\to\infty}\frac{e^{-D_k^w/S_k^w}}{1-\delta_k} = 0$.

In the following, we describe equilibrium when the exit rate is small. We consider the case in which $d^L < 1 < d^H$. In that case, buyers are on the short side in the low state and on the long side in the high state. The cases in which $d^H < 1$ or $d^L > 1$ are less interesting because in these cases it is known whether the buyers or the sellers are on the short side of the market.

As stated in the previous lemma, the difference between the sizes of the market sides in the inflow is magnified in the stock. Therefore, in the low state, the number of buyers per seller vanishes to zero, and a buyer is almost sure to be the sole bidder. In the high state, the number of buyers per seller diverges to infinity, and there is almost never only a single bidder.

The fact that a buyer becomes sure to be the sole bidder in the low state has an immediate implication: If a buyer is the sole bidder, the buyer wins and pays nothing. Therefore, in the low state, payoffs must converge to v for all buyers. Consequently, the characterization of the limit trading outcome conditional on the low state is straightforward. The corresponding fact for the high state has no immediate implication, however. Even though a bidder becomes less and less likely to be the sole bidder, the bidder also becomes more and more patient. If the bidder becomes patient fast enough, the bidder could just wait to be the sole bidder and receive the good for free too. In the second part of (iib), we show that this is not the case. Relative to the buyer's increasing patience, the probability of being the sole bidder converges to zero even faster. Therefore, in the limit, a buyer who uses the strategy of always bidding zero would almost surely have to exit the market before being able to trade.[21]

Let us discuss equilibrium bid patterns. Lemma 7 implies that, conditional on the high state, buyers learn that the state is high quickly after entry, after losing only once. This is because losing is very unlikely in the low state, but it is very likely in the high state. However, optimal bids depend on the buyer's belief conditional on being pivotal (tied at the top). Therefore, we characterize beliefs conditional on being *pivotal*. As the next result shows, being pivotal is very good news, indicating that the low state is very likely. Intuitively, the growing imbalance of the two sides of the market implies a very strong winner's curse.

Let $\theta_k^t(\theta)$ denote the posterior of a buyer who has entered the market with a prior θ and who has lost t times. The following proposition characterizes $\theta_k^0\left(\theta_k^t(\theta), \theta_k^t(\theta)\right)$, the posterior conditional on being pivotal (tied at the top) after having lost t times before.

Proposition 3 (Time Pattern on Bids.) *Suppose that* $d^L < 1 < d^H$ *and suppose that* $\lim_{k\to\infty}(1-\delta_k) = 0$*. Let* $\theta_k^t = \theta_k^t(\theta)$ *for some* $\theta \in [\underline{\theta}, \overline{\theta}]$*.*
(i) For any t*, (a)* $\lim_{k\to\infty}\theta_k^0\left(\theta_k^t, \theta_k^t\right) = 0$*, and (b)* $\lim_{k\to\infty}\beta_k(\theta_k^t) = 0$*.*
(ii) Let $t_k \equiv \frac{-0.5}{(1-\delta_k)\ln(1-\delta_k)}$*. Then, (a)* $\lim_{k\to\infty}\theta_k^0(\theta_k^{t_k}, \theta_k^{t_k}) = 1$*, and (b)* $\lim_{k\to\infty}\beta_k(\theta_k^{t_k}) = v$*.*

The proposition is not part of the proof of Proposition 2. In fact, we use findings from the proof of Proposition 2 to prove it. We state the proposition because we believe it provides some interesting insights into bidding when the exit rate is small.

As discussed previously, $\lim \theta_k^t(\theta) = 1$ as $k \to \infty$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$ and for all $t \geq 1$; that is, bidders who have lost at least once have a posterior that puts probability

[21] The Lemma 7 may suggest that learning from the differential matching probabilities is critical for price discovery. In our extension, we allow for the number of buyers and sellers to be the same in both states and show that the possibility of price discovery is not affected.

close to one on $w = H$. The first part of the proposition states that, nevertheless, the event of being tied conditional on having belief θ_k^t is sufficiently "good news" such that the posterior switches to putting probability zero on the high state. Therefore, conditional on being tied, a buyer believes that the continuation payoff is close to v and, consequently, bids zero. Two related and immediate consequences of Part (i) of Proposition 3 are that when $(1 - \delta_k) \to 0$, (a), fixing any time t, the bid of a buyer who has entered t periods ago *decreases* to zero when $(1 - \delta_k)$ decreases, and, (b), buyers bid close to zero for an increasingly long time. This observation illustrates why proving convergence to the competitive outcome is not immediate from the fact that the imbalance of the masses of buyers and sellers in the stock explodes in the high state.

Of course, the finding from Proposition 2 requires that buyers stop bidding zero at some time and bid close to v eventually. This is reflected in Part (ii) of Proposition 3. The significance of that part is that the number t_k is chosen so that $\lim (\delta_k)^{t_k} = 1$ as $k \to \infty$: the probability of exogenous exit within t_k periods is vanishing to zero. Part (ii) states that after at most t_k periods buyers are eventually sufficiently pessimistic that they bid high. The fact that $\lim (\delta_k)^{t_k} = 1$ can be interpreted as saying that buyers start bidding high "quickly" relative to the exit rate $(1 - \delta_k)$.

Proposition 3 illustrates the combined effect of the winner's and the loser's curse. Initial bids are predominantly shaped by the *winner's curse* (buyers bid cautiously low to avoid winning in the low state). Eventually, however, the *loser's curse* is sufficiently strong so that after losing at most t_k number of periods, buyers bid close to their maximum willingness to pay.

Rates of Convergence. The presence of aggregate uncertainty affects bidding, and, consequently, expected prices, as illustrated by Proposition 3. In general, the presence of aggregate uncertainty may imply that prices are further away from market-clearing, that is, further below v in the high state and higher above zero in the low state than prices would if the state were known. To isolate the "price distortion" that results from aggregate uncertainty, we compare the expected prices from our model with aggregate uncertainty to the equilibrium prices when the state is known, keeping the trading environment unchanged otherwise. We then study the effect of aggregate uncertainty on the level of price distortions by comparing the rate of convergence with and without uncertainty.

The model without uncertainty is introduced in the online Appendix. We characterize the equilibrium outcome for a sequence of survival rates $\{\delta_k\}$, $\delta_k \to 1$, for each state. Without uncertainty, all buyers have the same continuation payoffs in our model, and, hence, bid the same. This makes it straightforward to solve the equilibrium explicitly and to derive the expected trading price, which we denote by

$P^w_{C,k}$. The subscript C indicates the complete information variant. We then compare the price without uncertainty to the expected price from our base model, defined as

$$P^w_{I,k} \equiv \int_{\underline{\theta}}^{\bar{\theta}} P^w_k(\theta) dG^w(\theta),$$

where the subscript I indicates the (original) incomplete information variant. For the following proposition, note that $\lim_{k\to\infty}(1-\delta_k)\, e^{\left(d^H-1\right)(1-\delta_k)^{-1}} = \infty$.

Proposition 4 (Rates of Convergence.) *Suppose $d^H > 1 > d^L$. Then,*

$$\begin{aligned}
\lim_{k\to\infty} \frac{\frac{v-P^H_{I,k}}{v-P^H_{C,k}}}{(1-\delta_k)\, e^{(d^H-1)(1-\delta_k)^{-1}}} &> 0, \\
\lim_{k\to\infty} \frac{P^L_{I,k}}{P^L_{C,k}} &= 1.
\end{aligned}$$

To interpret the proposition, recall that the competitive price is v if $w = H$ and that the competitive price is zero if $w = L$. Proposition 2 and our analysis in the Appendix imply that the expected price converges to the competitive price in either state. However, when $1 - \delta_k$ is close to zero, the proposition implies that the expected price is "infinitely" further away from v in the high state when there is aggregate uncertainty than when there is no uncertainty. In the low state, the expected price is essentially the same whether or not there is aggregate uncertainty when $1 - \delta_k$ is close to zero.

The first part of the proposition illustrates the price impact of the bid-shading identified in Proposition 3. Common bid-shading decreases the rate of convergence substantially in the high state. In the low state, bid-shading does not create price distortions, and convergence is at the same rate.

The asymmetric finding for the states is likely due to the asymmetric treatment of buyers and sellers in our base model. In the following Section, we allow sellers to set reserve prices. There, sellers may resist trading at low prices by setting high reserve prices for a long time, introducing a problem that is symmetric to the bid-shading of the buyers. However, we cannot characterize equilibrium bidding explicitly in the following model, and we do not know the effect of aggregate uncertainty on the relative rate of convergence in that model.

6 Heterogeneous Buyers and Strategic Sellers

Two assumptions make our base model tractable: buyers have one-dimensional types (beliefs) and sellers do not take actions. We now lift these restrictions. Buyers have heterogeneous valuations and sellers can set a reserve price for their auctions. This extension gives rise to a richer economic environment. First, with heterogeneous preferences, whether trade takes place at the correct prices is consequential for efficiency.[22] Second, with strategic agents on both sides of the market, the market may turn into a two-sided war of attrition, with buyers insisting on low prices (bidding low for many periods) and sellers insisting on high prices (setting high reserve prices).

However, without the simplifying assumption from the base model, buyers' types are now multidimensional, so that we can no longer use standard techniques from auction theory to characterize bidding. In addition, with both market sides taking actions, we can no longer avoid the multiplicity problem that commonly plague models of bargaining with two-sided asymmetric information with interdependent preferences. Because of these problems, we cannot characterize equilibrium explicitly or prove its existence.

We now extend our base model. The mass of buyers who enter the market is either d^L or d^H, where $d^L \leq d^H$, so that we allow the number of buyers to be the same. Buyers' valuations are now drawn from the unit interval $[0, 1]$. The cumulative distribution function of the valuations of entering buyers in state w is $F^w(v)$. The functions F^H and F^L are absolutely continuous and have full support. The mass of the entering sellers is one in either state. We consider only the case where $d^H \geq d^L > 1$.

The market-clearing price in state w, denoted p_*^w, satisfies

$$1 = d^w \left(1 - F^w \left(p_*^w\right)\right).$$

We assume that $p_*^H > p_*^L$. Since F^L has full support, $p_*^L > 0$. In the competitive outcome relative to the quasilinear economy defined by the population of entering buyers and sellers, buyers with valuations above p_*^w receive the good and pay p_*^w, while buyers with valuations below do not. All sellers trade their good at p_*^w.

Each entering buyer privately observes a signal about the state. We call the combination of signals and valuations, $(\theta, v) \in [0, 1]^2$, a buyer's type. We assume that valuations and signals are independently and identically distributed conditional on the state. The joint distribution of the Bayesian posteriors and values induced by the signal is given by the cumulative distribution function G_B^w. Sellers, too, receive signals. The cumulative distribution of posteriors for sellers is G_S^w. We assume that G_B^w and G_S^w are absolutely continuous. Thus, almost no trader knows

[22] In our base model, whether or not prices aggregate information has no consequence for welfare. Our base model shares this feature with many of the standard models of information aggregation in large common value auctions; see, e.g., Milgrom (1979) and Pesendorfer and Swinkels (1997).

the state initially and we allow the initial distribution of beliefs to have arbitrarily small support. Bayesian updating requires that the distribution of beliefs satisfies no-introspection.[23]

As before, buyers are matched to sellers according to a Poisson distribution with a parameter equal to the ratio of the number of buyers to sellers. Each seller runs a second-price auction among the matched buyers. Different from before, sellers set a reserve price before buyers bid. The reserve price is observable to the buyers the seller is matched with. The buyers and the seller do not observe the number of bidders at an auction (neither when they choose their bids and the reserve price, nor afterwards) and the bids are not observed afterwards by the losing bidders. A buyer who bids strictly below the reserve price is said to not participate in the auction and the seller does not observe buyers who do not participate. If there is at least one participating buyer, then the seller exchanges the good with the highest bidder for a price that is equal to either the second highest bid among participating buyers or the reserve price if only one buyer participates. Ties among buyers are broken randomly.

We consider steady-state equilibria. Equilibria consist of a (Markovian and pure) bidding strategy $\beta(\theta, v)$ and a (Markovian and pure) reserve price strategy $\rho(\theta)$. There is a steady-state population, characterized by the mass of buyers and sellers, D^w and S^w, and a probability measure on buyers' types (valuations and beliefs) and sellers' types (beliefs), denoted by Γ_B^w and Γ_S^w, respectively. The interim belief of a buyer who observes a reserve price r with prior θ is denoted $\theta^I(\theta, r)$. The updated belief of a buyer with interim belief θ^I, conditional on losing with bid b given a reserve price r is denoted by $\theta_B^+\left(\theta^I, r, b\right)$. The updated belief of a seller with prior belief θ conditional on not selling with reserve price r is denoted by $\theta_S^+(\theta, r)$. The Appendix states the previous requirements explicitly and provides further formalization.

Let $Q_B^w(\theta, v)$ denote the lifetime trading probability in state w of a buyer having type (θ, v) and let $P_B^w(\theta, v)$ denote its expected price conditional on trading. Let $Q_S^w(\theta)$ and $P_S^w(\theta)$ denote the trading probability and trading price for a seller having type θ. In the competitive outcome for the high state, the buyers' trading probabilities are $Q_B^H(\theta, v) = 1$ for $v > p_*^H$ and $\theta > 0$, and $Q_B^H(\theta, v) = 0$ for $v < p_*^H$ and $\theta > 0$, the sellers' trading probabilities are $Q_S^H(\theta) = 1$ for $\theta > 0$, and the price is $P_B^H(\theta, v) = p_*^H$ for $v \geq p_*^H$ and $\theta > 0$, and $P_S^H(\theta) = p_*^H$ for $\theta > 0$. The competitive outcome for a trader who is mistakenly convinced of the low state is not defined, as is the outcome for a marginal buyer with valuation $v = p_*^H$. The competitive

[23] That is, $\theta/(1-\theta) = g_S^H(\theta)/g_S^L(\theta)$ for all θ and $\theta/(1-\theta) = d^H g_B^H(\theta, v)/d^L g_B^L(\theta, v)$ for all (θ, v).

outcome for the low state is defined analogously.

We restrict attention to equilibria in *monotone strategies*, by which we mean that the reserve price $\rho(\theta)$ is *weakly* increasing in the prior and that the bid $\beta\left(\theta^I, v, r\right)$ is *weakly* increasing in the interim belief, the valuation, and the observed reserve price. Furthermore, we assume that the endogenous probability measures Γ_B^w and Γ_S^w are absolutely continuous and that its densities γ_B^w and γ_S^w satisfy no-introspection.[24]

Because the reserve price is observable to the buyers, it becomes a signal of the seller's beliefs about the state of the market, and there is freedom in assigning beliefs following off-equilibrium reserve prices. This freedom can then be used to support multiple equilibria.[25] To deal with this multiplicity, we assume that buyer's beliefs are *passive*. Specifically, $\theta^I(\theta, r) = \theta$ whenever $r \neq \rho\left(\theta'\right)$ for any θ': After observing a reserve price that is off the equilibrium path, the buyer's interim belief is equal to its prior. We return to a discussion of passive beliefs at the end of this section.

Taken together, a list $(\beta, \rho, S^w, D^w, \Gamma_S^w, \Gamma_B^w, \theta_S^+, \theta_B^+, \theta^I)$ is a *monotone steady-state equilibrium with passive beliefs* if the stock $(S^w, D^w, \Gamma_S^w, \Gamma_B^w)$ constitutes a steady-state given the bargaining strategies, strategies β and ρ are sequentially rational and monotone, beliefs θ_S^+, θ_B^+, and θ^I are consistent with Bayes' rule and off-equilibrium beliefs are passive. We refer to such a list as an *equilibrium* in what follows.

Given a sequence of vanishing exit rates $\{1 - \delta_k\} \to 0$, suppose there is a sequence of equilibria that gives rise to a sequence of outcomes, that is, trading probabilities and prices. We say that the sequence of outcomes converges to the competitive outcome if the trading probabilities and prices converge to the competitive outcome.

Proposition 5 *Consider an economy with heterogeneous buyers and strategic sellers. For every sequence of vanishing exit rates and for every sequence of monotone steady-state equilibria with passive beliefs, the sequence of trading outcomes converges to the competitive outcome for each state of nature.*

The proposition is proven in the Appendix in Section 9. The proof builds on methods from Lauermann (2012). In the following, we discuss the difficulties posed by signaling and the role of passive beliefs in overcoming those difficulties.

The proof method suggested in Lauermann (2012) is to show that there are no unrealized gains from trade in the limit between types of buyers and sellers that are

[24] That is, $\theta/(1-\theta) = S^H \gamma_S^H(\theta) / S^L \gamma_S^L(\theta)$ for all θ and $\theta/(1-\theta) = D^H \gamma_B^H(\theta, v) / D^L \gamma_B^L(\theta, v)$ for all (θ, v).

[25] Note that we cannot avoid this problem by assuming that the reserve price is secret or set only after the bids are made. With these assumptions, the buyers' bids would become signals about their beliefs, leading to similar problems.

matched frequently. Otherwise, a seller could keep offering a price at which exchange is mutually beneficial and be sure to be matched eventually to a buyer who accepts the offer (or vice versa). This observation is then used to establish that trading outcomes are pairwise efficient, and, hence, competitive. In the current context, this method of proof can be upset by the freedom in assigning off-equilibrium beliefs: Suppose that there is a set of types of buyers and sellers who make up a positive share of the stock even in the limit, and, hence, continue to be matched with positive probability. Suppose also that there is a price such that, at their current beliefs, trading at this price would be profitable for all these types of buyers and sellers. The problem is that the mutually agreeable price might be off the equilibrium path. Specifically, if a seller deviates from the equilibrium and offers the price nevertheless, the buyers interpret this price as information about the seller's belief about the state of nature. In particular, buyers may interpret the price offer as signaling that the seller is sure that the state of nature is such that the buyers can receive even better prices in the future. Consequently, the buyers refuse to trade. Thus, it may not be possible for the seller to profit from offering the price and gains from trade may remain unrealized in the limit. The assumption that beliefs are passive prevents this possibility.[26]

If beliefs are not passive, we believe that it is possible to construct non-competitive limit outcomes using the freedom in assigning off-equilbrium beliefs. However, we have not been able to construct such a non-competitive equilibrium. Constructing a non-competitive equilibrium would be useful because it would illustrate a particular problem that is introduced by aggregate uncertainty and that makes it harder for a market to reach the competitive outcome.

7 Discussion and Conclusion

Bid Disclosure.—In our model, learning is "minimal": losing buyers learn nothing except that they lost. In our companion paper, Lauermann and Virag (2012), we ask whether such nontransparent auctions would arise if each seller could individually choose the auction format. We show that sellers have an incentive to hide information from the buyers because of a "continuation value effect": If bidders receive information when losing, then they can refine their future bids, which raises the expected value of their outside options. This leads to less aggressive bidding and lower revenues for the seller. Countervailing the continuation value effect is the well-known linkage principle effect for common value auctions. We study how

[26] In our proof, the central application of passive beliefs is in Step 7 of Claim 6 on Page 53. There, we use this assumption to argue that the seller can profitably lower the reservation price. Trading at the lower price implies strictly higher profits than continued search for all types of sellers.

these two effects determine the sellers' preferences for information disclosure. For example, we show that the sellers do not have an incentive to reveal any information about the submitted bids after the auction.

Costly Reserve Prices in Base Model.—In our base model, the sellers do not set a reserve price. However, observe that individual sellers do not have a strong incentive to set reserve prices when the exit rate is small, in the sense that the expected winning bid is already close to the buyers' maximum willingness to pay in each state, given the continuation payoffs. We can formalize this observation: Suppose sellers in the base model have the option to set a reserve price at a small cost. Then, for a sufficiently small exit rate, it would be an equilibrium for sellers to not use that option. Given that no seller sets a positive reserve price, the resulting equilibrium outcome would then be equivalent to the equilibrium outcome of our original model.[27] In fact, we have verified that with costly reserve prices, all equilibria are competitive in the limit. Thus, adding a small cost of using a reserve price can substitute for the refinement of passive beliefs. Such a cost may not be unreasonable, as they may represent general hassle costs connected with using a reserve price.

Conclusion.—We have introduced a new framework to study price discovery through trading in a decentralized market. In our model, buyers learn about the relative scarcity of a good through repeated bidding in auctions. In particular, individual traders never observe the whole market and they directly interact only with small groups of traders. We characterized the resulting distribution of beliefs in the population, the learning process, and the bidding behavior of buyers in our base model. Despite the fact that there is no centralized price formation mechanism, we found that the equilibrium trading outcome is approximately Walrasian when the exit rate is small and search becomes cheap. Thus, prices reveal aggregate scarcity and correctly reflect economic value. However, comparing the outcome with and without uncertainty, we found that the Walrasian outcome is a better approximation when aggregate scarcity is known. This demonstrates the specific difficulties for markets to clear when demand and supply are unknown. We also demonstrated in an extension that convergence does not depend on the simplifications introduced in the base models. In particular, the trading outcomes converge even if there is heterogeneity and sellers choose their reserve prices strategically.

In this paper, we have used our model to study the possibility of price discovery. However, our model has broader applicability because it provides a tractable equilibrium framework for studying the search behavior of agents who learn about the

[27] This argument depends, of course, on the cost of setting a reservation price being large relative to the exit rate.

economy.[28] Search theory has been remarkably successful over the last decades but little progress has been made in incorporating the possibility of learning. Instead, models of search usually assume that the searching agents know the distribution of prices (wages, interest rates, etc.) that they are sampling from.[29] Our model may be a first step toward relaxing this assumption. Incorporating learning into models of search promises new insights into a broad range of economic phenomena.

[28] In the base model, equilibrium is explicitly computable and amenable to comparative static exercises.

[29] Rothschild (1974) already argued that "the results [from search theory] depend on the untenable assumption that searchers know the probability distribution from which they are searching," (p. 689) and "it seems absurd to suppose that consumers know them [the price distributions] with any reasonable degree of accuracy" (p692).

8 Appendix A

The first part of the Appendix contains the proofs of our main results from Section 5 (Price Discovery with Small Frictions). The second part of the Appendix contains the proof of our result from Section 6 (Heterogeneous Buyers and Strategic Sellers).

Most proofs of results from Section 4 (Characterization and Existence of Equilibrium) are contained in an **Online Appendix**.[30] Exceptions are the proofs of Lemma 1 (Unique Masses), Lemma 5 (Envelope Theorem) and Lemma 6 (Characterizing the Equilibrium Bidding) which we give here, too. We keep these three proofs in the regular Appendix because we use elements from these proofs when showing convergence and because the characterization of the equilibrium bids is a cornerstone of our convergence proof. The Online Appendix contains also the characterization of the speed of convergence.

8.1 Proof of Lemma 1 (Uniqueness of Masses)

We show that the steady-state conditions for the stocks can be written as:

$$d^w = (1-\delta)\,D^w + \delta S^w\left(1 - e^{-D^w/S^w}\right) \tag{11}$$

$$1 = (1-\delta)\,S^w + \delta S^w\left(1 - e^{-D^w/S^w}\right). \tag{12}$$

These conditions have a simple interpretation: the left-hand side is the inflow for each market side. The right-hand side is outflow from each market side, that is, the sum of the number of traders who exit through discouragement and the number of traders who exit through trade. The number of traders who exit through trade is $S^w\left(1-e^{-D^w/S^w}\right)$, which is equal for both market sides.

Rewriting the steady-state condition for buyers, (4),

$$\begin{aligned} D^w &= d^w + \delta D^w \int_0^1 \left(1 - e^{-D^w(1-\Gamma^w(\theta))/S^w}\right) d\Gamma^w(\theta) \\ &= d^w + \delta D^w - \delta \int_0^1 \frac{\partial}{\partial\theta}\left(S^w e^{-D^w(1-\Gamma^w(\theta))/S^w}\right) d\theta \\ &= d^w + \delta D^w - \delta S^w\left(1 - e^{-D^w/S^w}\right). \end{aligned}$$

Recall the steady-state condition for sellers, (3), $S^w = 1 + \delta S^w e^{-D^w/S^w}$. Rewriting the steady-state conditions further yields (11) and (12).

A solution to the steady-state conditions exists, and the solution is unique. The

[30] At http://sites.google.com/site/slauerma/Price-Discovery-Online-Appendix.pdf

difference of (11) and (12),

$$d^w - 1 = (1-\delta)(D^w - S^w), \tag{13}$$

defines D^w as a function of S^w, $J(S^w)$. We can write (12) as a function of S^w only,

$$1 = (1-\delta) S^w + \delta S^w \left(1 - e^{-J(S^w)/S^w}\right).$$

This equation has a solution by the intermediate value theorem. At $S^w \to 0$, the right-hand side becomes zero, while for $S^w \to \infty$, the right-hand side becomes infinite (recall that $\left(1 - e^{-J(S^w)/S^w}\right) \in [0,1]$). A solution exists in between.

The solution is unique. Let S'^w, D'^w and S''^w,D''^w be two solutions, and suppose that $D''^w \geq D'^w$. Then, by (13), $S''^w \geq S'^w$. We can show that (12) and $S''^w > S'^w$ leads to a contradiction. Hence, it must be that $S''^w = S'^w$, which implies $D''^w = D'^w$ by (13). The contradiction arises as follows. The first term of (12) is trivially strictly increasing in S^w. The second term (12) is also increasing in S^w and in D^w, which can be seen by inspection of the derivatives,[31]

$$\begin{aligned}
\frac{\partial}{\partial S^w}\left(S^w\left(1 - e^{-D^w/S^w}\right)\right) &= \left(1 - e^{-D^w/S^w}\right) - \frac{D^w}{S^w} e^{-D^w/S^w} \geq 0 \\
\frac{\partial}{\partial D^w}\left(S^w\left(1 - e^{-D^w/S^w}\right)\right) &= e^{-D^w/S^w} > 0.
\end{aligned}$$

Thus, if (12) holds for S'^w, it cannot also hold for $S''^w > S'^w$. Intuitively, if the number of buyers and sellers is higher, then (i) more sellers exit due to discouragement (the first term) and (ii) more sellers trade (the second term) because there are simply more sellers (the first derivative is positive) and, in addition, the number of buyers increases and so less sellers have no bidders (the second derivative is positive).

By assumption, $d^H > d^L$. We show that this implies $D^H > D^L$ and $S^L < S^H$. First, it cannot be that both the number of sellers and the number of buyers increases (or decreases) when the state is changed from L to H. By our earlier observation, if both, the number of sellers and buyers increases, then the right-hand side of (12) would strictly increase, leading to a failure of the equation. Similarly, the right-hand side of (12) would strictly decrease if both market sides shrink. Because $d^H > d^L$, inspection of (13) shows that the difference $\left(D^H - S^H\right) > \left(D^L - S^L\right)$. Hence, it cannot be that the buyers' market side weakly decreases while the sellers' market side weakly increases. Therefore, $D^H > D^L$ and $S^L < S^H$, as claimed. □

[31] Let $x = D^w/S^w$. The first inequality follows if $\frac{1}{x}\left(1 - e^{-x}\right) \geq e^{-x}$. This inequality holds if $e^x - 1 \geq x$, which is true, since, at zero, $e^0 - 1 \geq 0$, and for $x > 0$, $(e^x - 1 - x)' = e^x - 1 \geq 0$.

8.2 Proof of Lemma 5 (Envelope Theorem)

Convexity follows from a standard argument. By definition, $V(\theta) = EU(\theta, \beta|\theta)$. With $\theta^{\alpha} = (\alpha\theta' + (1-\alpha)\theta'')$,

$$
\begin{aligned}
V(\alpha\theta' + (1-\alpha)\theta'') &= \alpha EU\left(\theta^{\alpha}, \beta|\theta'\right) + (1-\alpha) EU\left(\theta^{\alpha}, \beta|\theta''\right) \\
&\leq \alpha EU\left(\theta', \beta|\theta'\right) + (1-\alpha) EU\left(\theta'', \beta|\theta''\right) = \alpha V\left(\theta'\right) + (1-\alpha) V\left(\theta''\right).
\end{aligned}
$$

The equalities follow by definition of EU and V and linearity of EU. The inequality follows from optimality of β.

The envelope formula follows from standard arguments as well: (i) optimality requires $EU(\theta, \beta|\hat{\theta}) \leq EU(\hat{\theta}, \beta|\hat{\theta})$ for all $\theta, \hat{\theta}$ and (ii) $EU(\theta, \beta|\hat{\theta})$ is differentiable everywhere in $\hat{\theta}$. Hence, Theorem 1 by Milgrom and Segal (2002) implies $V'(\theta) = \frac{\partial}{\partial\hat{\theta}} EU(\theta, \beta|\hat{\theta})|_{\hat{\theta}=\theta}$ at interior differentiable points of V. Linearity of $EU(\theta, \beta|\hat{\theta})$ in $\hat{\theta}$ implies that $EU(\theta, \beta|\hat{\theta}) = EU(\theta, \beta|\theta) + (\hat{\theta} - \theta)\frac{\partial EU(\theta,\beta|\hat{\theta})}{\partial\hat{\theta}}|_{\hat{\theta}=\theta}$. Together, (9) follows. *QED.*

8.3 Proof of Lemma 6 (Equilibrium Candidate).

The derivative of the objective function (5) is

$$
\beta^{-1\prime}(\beta(x))(\gamma^{\theta}_{(1)}(v - \beta(x) - \delta V\left(\theta^{+}(x,\theta)\right)) + \delta(1 - \Gamma^{\theta}_{(1)}(x))V'\left(\theta^{+}(x,\theta)\right)\frac{\partial\theta^{+}(x,\theta)}{\partial x}). \tag{14}
$$

The derivative exists for almost every type in the support of Γ^{w}. The optimal bid for almost all types is characterized by the first-order condition (14)= 0.

Note that

$$
\frac{\partial\theta^{+}(x,\theta)}{\partial x} = \frac{-\theta\gamma^{H}_{(1)}(x)(1 - \Gamma^{\theta}_{(1)}(x)) + \gamma^{\theta}_{(1)}(x)\theta(1 - \Gamma^{H}_{(1)}(x))}{(1 - \Gamma^{\theta}_{(1)}(x))^{2}}. \tag{15}
$$

Further,

$$
\frac{\gamma^{\theta}_{(1)}(x)\theta(1 - \Gamma^{H}_{(1)}(x))}{1 - \Gamma^{\theta}_{(1)}(x)} - \theta\gamma^{H}_{(1)}(x) = \gamma^{\theta}_{(1)}(x)\left(\theta^{+}(x,\theta) - \theta^{0}(x,\theta)\right) \tag{16}
$$

by the definitions of θ^{+} and $\theta^{0}(x,\theta)$. Using (15) and (16), the necessary first-order condition (14)= 0 can be rewritten as (10). *QED*

8.4 Proof of Proposition 2: Price Discovery

The proposition follows from a sequence of lemmas. We start by proving **Lemma 7**. Recall Equation (13),

$$S_k^w = D_k^w - \frac{d^w - 1}{1 - \delta_k}. \tag{17}$$

Substituting (17) into (12) yields

$$1 - e^{-D_k^w / S_k^w} = \frac{1 - (1 - \delta_k)\, S_k^w}{\delta_k S_k^w} = 1 + \frac{1 - S_k^w}{\delta_k S_k^w}. \tag{18}$$

We can solve this equation for D_k^w to obtain

$$D_k^w = -S_k^w \ln \frac{S_k^w - 1}{\delta_k S_k^w}. \tag{19}$$

Case 1: $d^w < 1$. In this case, $\lim \frac{d^w - 1}{1 - \delta_k} = -\infty$, so that (17) implies $\lim (S_k^w - D_k^w) = \infty$; hence, $\lim S_k^w = \infty$. This implies that the right–most side of (18) converges to 0. Therefore, the limit of the left-most side $\lim \left(1 - e^{-D_k^w / S_k^w}\right) = 0$, that is,

$$\lim_{k \to \infty} \mu_k^w = \lim_{k \to \infty} D_k^w / S_k^w = 0,$$

as claimed. From the steady-state condition for buyers, (4), $D_k^w \geq d^w$. Reordering terms and evaluating the integral on the right-hand side of (4) at zero,

$$d^w \geq (1 - \delta_k)\, D_k^w + \delta_k D_k^w \Gamma_{k,(1)}^w (0) = (1 - \delta_k)\, D_k^w + \delta_k D_k^w e^{-\mu_k}.$$

Taking limits on the last two inequalities implies

$$\lim_{k \to \infty} D_k^w \geq d^w \geq \lim_{k \to \infty} D_k^w e^{-\mu_k} = \lim_{k \to \infty} D_k^w.$$

Hence, $\lim D_k^w = d^w$, as claimed. Therefore, from (17) it follows that

$$\lim_{k \to \infty} (1 - \delta_k) S_k^w = 1 - d^w. \tag{20}$$

Letting $\mu_k^w = D_k^w / S_k^w$, it follows from (19) and (20) that

$$\lim_{k \to \infty} \frac{\mu_k^w}{1 - \delta_k} = \frac{d^w}{1 - d^w}. \tag{21}$$

Case 2: $d^w > 1$. From (17), it follows that $\lim D_k^w - S_k^w = \infty$; thus, $\lim D_k^w = \infty$. Then, (19) implies that $\lim S_k^w = 1$.[32] This implies that $\mu_k^w \to \infty$. The rest of the

[32] To see this, note that if $\lim S_k^w > 1$, but finite was true, then (19) would imply that $\lim D_k^w < \infty$,

proof establishes that $\lim \frac{1-\delta_k}{e^{-\mu_k^w}} = \infty$. Using (17) and $\lim S_k^w = 1$ yields that

$$\lim_{k\to\infty}(1-\delta_k)D_k^w = d^w - 1. \tag{22}$$

Formula (22) and $\lim S_k^w = 1$ imply that

$$\lim_{k\to\infty}(1-\delta_k)\mu_k^w = d^w - 1. \tag{23}$$

Finally, using (23) implies that $\lim \frac{1-\delta_k}{e^{-\mu_k^w}} = \lim(1-\delta_k)e^{\mu_k^w} = (d^w - 1)\lim \frac{e^{\mu_k^w}}{\mu_k^w} = \infty$.
□

Let $Q_k^w(\theta)$ denote the lifetime trading probability of a buyer having type θ,

$$Q_k^w(\theta) = (1-\xi_k^w(\theta)) + \delta_k \xi_k^w(\theta)\left(1-\xi_k^w\left(\theta_k^1(\theta)\right)\right) + \delta_k^2 \xi_k^w(\theta)\,\xi_k^w\left(\theta_k^1(\theta)\right)\left(1-\xi_k^w\left(\theta_k^2(\theta)\right)\right) + \cdots$$

where $1-\xi_k^w(\theta)$ denotes the probability that a buyer with type θ trades in any given period, and $\theta_k^t(\theta)$ denotes the posterior of a buyer with a prior θ who has lost t times. The steady-state conditions imply a bound on the average expected trading probability:

Lemma 8 *The average expected lifetime trading probability is bounded by the ratio of the number of entering sellers to the number of entering buyers,* $\int_{\underline{\theta}}^{\overline{\theta}} Q_k^w(\theta)\, dG^w(\theta) \le 1/d^w$. *Moreover, if* $1 < d^w$, *then the average expected trading probability* $\lim_{k\to\infty} \int_{\underline{\theta}}^{\overline{\theta}} Q_k^w(\theta)\, dG^w(\theta) = 1/d^w$.

Proof: Let $\left(\theta_k^t\right)^{-1}(\theta')$ be the generalized inverse, $\left(\theta_k^t\right)^{-1}(\theta') = sup\left\{\theta | \theta_k^t(\theta) \le \theta'\right\}$. The steady-state conditions require that for every θ

$$\begin{aligned} D_k^w \Gamma_k^w(\theta) = \quad & d^w\Big(\int_0^\theta dG^w(\tau) + \delta_k \int_0^{\left(\theta_k^1\right)^{-1}(\theta)} \xi_k^w(\tau)\, dG^w(\tau) \\ & + \delta_k^2 \int_0^{\left(\theta_k^2\right)^{-1}(\theta)} \xi_k^w(\tau)\,\xi_k^w\left(\theta_k^1(\tau)\right) dG^w(\tau) + \cdots. \end{aligned} \tag{24}$$

By the fundamental theorem of calculus for Lebesgue integration, we can multiply the above identity with $1-\xi_k^w(\tau)$ point-by-point, which yields

---

a contradiction with what we have already established above. If $\lim S_k^w = \infty$, then $\log \frac{S_k^w - 1}{\delta_k S_k^w} \to 0$, and therefore, by (19), $\lim \mu_k^w = \lim D_k^w / S_k^w = 0$, which contradicts $\lim D_k^w - S_k^w = \infty > 0$. Therefore, $\lim S_k^w = 1$ must hold.

$$
\begin{aligned}
& D_k^w \int_0^\theta \left(1-\xi_k^w(\tau)\right) d\Gamma_k^w(\tau) \\
= \; & d^w \Big(\int_0^\theta \left(1-\xi_k^w(\tau)\right) dG^w(\tau) + \delta_k \int_0^{\left(\theta_k^1\right)^{-1}(\theta)} \left(1-\xi_k^w\left(\theta_k^1(\tau)\right)\right) \xi_k^w(\tau)\, dG^w(\tau) + \ldots
\end{aligned}
$$

Evaluating at $\theta = 1$ and using the definition of $Q_k^w(\theta)$ to simplify the right-hand side

$$
\begin{aligned}
D_k^w \int_0^1 \left(1-\xi_k^w(\tau)\right) d\Gamma_k^w(\tau) &= d^w \Big(\int_0^1 \big(\left(1-\xi_k^w(\tau)\right) + \delta_k \left(1-\xi_k^w\left(\theta_k^1(\tau)\right)\right) \xi_k^w(\tau) \\
&\quad + \delta_k^2 \left(1-\xi_k^w\left(\theta_k^2(\tau)\right)\right) \xi_k^w(\tau)\, \xi_k^w\left(\theta_k^1(\tau)\right) + \ldots \big) dG^w(\tau) \\
&= d^w \int_0^1 Q_k^w(\theta)\, dG^w(\theta).
\end{aligned} \tag{25}
$$

As shown in the proof of Lemma 1, $D_k^w \int_0^1 \left(1-\xi_k^w(\tau)\right) d\Gamma_k^w(\tau) = S_k^w\left(1-e^{-D_k^w/S_k^w}\right)$, the total mass of buyers who trade in any period is equal to the total mass of sellers who trade. Rewriting the steady-state condition for the sellers, (3), implies $1 = S_k^w\left(1-\delta_k e^{-D_k^w/S_k^w}\right)$. Because $1 \geq \delta_k$, $1 \geq S_k^w\left(1-e^{-D_k^w/S_k^w}\right)$. Taken together, we have shown the following chain of (in-)equalities, which proves the first claim of the lemma:

$$
1 \geq S_k^w\left(1-e^{-D_k^w/S_k^w}\right) = D_k^w \int_0^1 \left(1-\xi_k^w(\tau)\right) d\Gamma_k^w(\tau) = d^w \int_0^1 Q_k^w(\theta)\, dG^w(\theta). \tag{26}
$$

Equation (12) implies that if $d^w > 1$, then $S_k^w\left(1-e^{-D_k^w/S_k^w}\right) \to 1$. Taking limits on the last three equalities in (26) implies the second claim of the lemma,

$$
1 = \lim_{k\to\infty} S_k^w\left(1-e^{-D_k^w/S_k^w}\right) = \lim_{k\to\infty} d^w \int_0^1 Q_k^w(\theta)\, dG^w(\theta).
$$

□

The following Lemma strengthens the finding of Lemma 8 for a special case.

Lemma 9 *Suppose that $d^H > 1$ and $d^L < 1$. Then, for all $\theta \in [\underline{\theta}, \overline{\theta}]$*

$$
\lim_{k\to\infty} Q_k^H(\theta) = 1/d^H.
$$

Proof: First, the trading probability is monotone in the type: Let $\theta_l < \theta_h$ and let $\{\theta_l^t\}_{t=0}^\infty$ and $\{\theta_h^t\}_{t=0}^\infty$ be the sequence of updated beliefs after losing t times. By monotonicity of θ_k^+, $\theta_l^t < \theta_h^t$ for all t; hence, $\beta_k\left(\theta_l^t\right) < \beta\left(\theta_h^t\right)$ for all t. Therefore, the probability of winning in any given period after having lost t times, $1-\xi_k^w\left(\theta_l^t\right) <$

$1-\xi_k^w\left(\theta_h^t\right)$ for all t; hence,

$$
\begin{aligned}
Q_k^w\left(\theta_l\right) &= \left(1-\xi_k^w\left(\theta_l^0\right)\right)+\delta_k \xi_k^w\left(\theta_l^0\right)\left(1-\xi_k^w\left(\theta_l^1\right)\right)+\delta_k^2 \xi_k^w\left(\theta_l^1\right) \xi_k^w\left(\left(\theta_l^0\right)\right) \ldots . \\
<Q_k^w\left(\theta_h\right) &= \left(1-\xi_k^w\left(\theta_h^0\right)\right)+\delta_k \xi_k^w\left(\theta_h^0\right)\left(1-\xi_k^w\left(\theta_h^1\right)\right)+\delta_k^2 \xi_k^w\left(\theta_h^1\right) \xi_k^w\left(\left(\theta_h^0\right)\right) \ldots . \quad .
\end{aligned}
$$

The posterior of the most optimistic new buyer after losing once becomes one, $\theta_k^1(\underline{\theta})=\theta_k^+(\underline{\theta}, \underline{\theta}) \rightarrow 1$, since the likelihood ratio of losing $\frac{1-e^{-D_k^H / S_k^H}}{1-e^{-D_k^L / S_k^L}} \rightarrow \infty$ by Lemma 7. This implies that $\lim \theta_k^+(\underline{\theta}, \underline{\theta})>\bar{\theta}$. Hence, by the monotonicity of the trading probability, we can "sandwich" the trading probability of all $\theta \in[\underline{\theta}, \bar{\theta}]$ for sufficiently large k,

$$
Q_k^H\left(\theta_k^+(\underline{\theta}, \underline{\theta})\right) \geq Q_k^H(\theta) \geq Q_k^H(\underline{\theta}) \quad \forall \quad \theta \in[\underline{\theta}, \bar{\theta}],\ k \text{ large.} \tag{27}
$$

Using (27) and Lemma 8,

$$
\begin{aligned}
& \liminf_{k \rightarrow \infty} \int_{\underline{\theta}}^{\bar{\theta}} Q_k^H\left(\theta_k^+(\underline{\theta}, \underline{\theta})\right) d G^H(\theta) \qquad (28)\\
\geq\ & \lim_{k \rightarrow \infty} \int_{\underline{\theta}}^{\bar{\theta}} Q_k^H(\theta)\, d G^H(\theta)=1 / d^H \geq \limsup_{k \rightarrow \infty} \int_{\underline{\theta}}^{\bar{\theta}} Q_k^H(\underline{\theta})\, d G^H(\theta) .
\end{aligned}
$$

By construction, $Q_k^H(\underline{\theta}) \geq \delta_k Q_k^H\left(\theta_k^+(\underline{\theta}, \underline{\theta})\right)$. By monotonicity of θ^+ and monotonicity of Q_k^H, $Q_k^H(\underline{\theta}) \leq Q_k^H\left(\theta_k^+(\underline{\theta}, \underline{\theta})\right)$. Therefore, the difference $Q_k^H\left(\theta_k^+(\underline{\theta}, \underline{\theta})\right)-Q_k^H(\underline{\theta}) \in\left[0,1-\delta_k\right]$. When $\delta_k \rightarrow 1$, $\lim \left(Q_k^H\left(\theta_k^+(\underline{\theta}, \underline{\theta})\right)-Q_k^H(\underline{\theta})\right)=0$ (the expected trading probability with the initial type $\underline{\theta}$ and the expected trading probability after updating once become the same). Hence, $\liminf_{k \rightarrow \infty} Q_k^H\left(\theta_k^+(\underline{\theta}, \underline{\theta})\right) \leq \limsup\ Q_k^H(\underline{\theta})$. This inequality together with the inequalities (28) implies

$$
\lim_{k \rightarrow \infty} Q_k^H\left(\theta_k^+(\underline{\theta}, \underline{\theta})\right)=1 / d^H=\lim_{k \rightarrow \infty}\ Q_k^H(\underline{\theta});
$$

(recall, $\int_{\underline{\theta}}^{\bar{\theta}} d G^H(\theta)=1$). Hence, (27) implies $\lim Q_k^H(\theta)=1 / d^H$ for all $\theta \in[\underline{\theta}, \bar{\theta}]$. $\square$

We prove that trading probabilities satisfy the conditions of Proposition 2.

Lemma 10 *Trading probabilities satisfy:*

$$
\lim_{k \rightarrow \infty} Q_k^w(\theta)=\begin{cases}1 & \text{if } d^w<1 \\ \frac{1}{d^w} & \text{if } d^w>1\end{cases} \quad \text{and} \quad \lim_{k \rightarrow \infty} Q_k^w(S)=\begin{cases}d^w & \text{if } d^w<1 \\ 1 & \text{if } d^w>1 .\end{cases}
$$

Proof: For buyers: If $d^w<1$, then $\lim_{k \rightarrow \infty} Q_k^w(\theta)=1$ is immediate from Lemma 7. We have argued that $\lim_{k \rightarrow \infty} Q_k^w(\theta)=\frac{1}{d^w}$ if $d^w>1$ for the case $w=H$ and $d^L<1$ in

Lemma 9. The case in which in both states $d^w > 1$ follows from the steady-state conditions along similar lines. We omit the proof of that case.

For sellers: The trading probability is recursively defined as $Q_k^w(S) = 1 - e^{-D_k^w/S_k^w} + \delta_k e^{-D_k^w/S_k^w} Q_k^w(S)$. If $d^w > 1$, then $\lim_{k\to\infty} Q_k^w(S) = 1$ follows from $D_k^w/S_k^w \to \infty$, shown in Lemma 7.

If $d^w < 1$, then (20) implies $(1-\delta_k)S_k^w = 1-d^w$. From the steady-state condition, (12),

$$1 = (1-\delta_k) S_k^w + \delta_k S_k^w \left(1 - e^{-D_k^w/S_k^w}\right).$$

Rewriting the definition of $Q_k^w(S)$, $1 - e^{-D_k^w/S_k^w} = \frac{(1-\delta_k)Q_k^w(S)}{1-\delta_k Q_k^w(S)}$, substituting into the steady-state condition, and taking limits,

$$1 = \lim (1-\delta_k) S_k^w + \lim \delta_k S_k^w \frac{(1-\delta_k) Q_k^w(S)}{1-\delta_k Q_k^w(S)} = 1 - d^w + (1-d^w) \lim \frac{Q_k^w(S)}{1-Q_k^w(S)},$$

from which $\lim Q_k^w(S) = d^w$ follows, as claimed. *QED.*

Let $Q_k^w(\theta, b')$ denote the probability that a type θ eventually ends up trading at a price $p \leq b'$, and let θ_k denote the highest type in the stock who bids below b', $\theta_k = sup\{\theta | \beta_k \leq b',\, \theta \in supp\Gamma_k^w\}$ (if there is no such type, $\theta_k = 0$). The probability $Q_k^w(\theta, b')$ is defined as

$$\begin{aligned} Q_k^w\left(\theta, b'\right) &= (1-\xi_k^w(\min\{\theta_k, \theta\})) + \delta_k (1-\xi_k^w(\theta))\, \xi_k^w\left(\min\left\{\theta_k, \theta_k^1(\theta)\right\}\right) \\ &\quad + \delta_k^2 (1-\xi_k^w(\theta)) \left(1-\xi_k^w\left(\theta_k^1(\theta)\right)\right) \xi_k^w\left(\min\left\{\theta_k, \theta_k^2(\theta)\right\}\right) + ..., \end{aligned}$$

where we need to use $(\min\{\theta_k, \theta\})$ because $\beta_k(\theta)$ might be below b'; that is, $\theta < \theta_k$.

Let $\overline{V} = \lim V_k(1)$. We show that the expected price conditional on winning becomes equal to $v - \bar{V}$. To show this, we want to prove that if a bidder wins with some bid with positive probability, the second highest bidder bids almost surely $\beta = v - \bar{V}$ (no bidder would bid higher). The proof works as follows: If there is a positive chance to win against a buyer with a belief θ', then the posterior of this buyer conditional on being tied must converge to one; formally,

$$\forall \left\{\theta_k'\right\}: \qquad \lim Q_k^H\left(\theta, \beta_k\left(\theta_k'\right)\right) > 0 \Rightarrow \lim \theta_k^0\left(\theta_k', \theta_k'\right) = 1.$$

Because $\theta_k^0\left(\theta_k', \theta_k'\right) \to 1$, the bid $\beta_k\left(\theta_k'\right)$ is shown to converge to $\lim V_k(1) = \bar{V}$.

Lemma 11 *Suppose that $d^H > 1$ and $d^L < 1$. Let $\overline{V} = \lim_{k\to\infty} V_k(1)$. For any type $\theta \in [\underline{\theta}, \overline{\theta}]$ the expected price conditional on winning in the high state converges to $v - \bar{V}$, $\lim_{k\to\infty} E_k[p, \theta | H] = v - \overline{V}$.*

Proof: Suppose that there are some b' and $\theta^* \in \left[\underline{\theta}, \bar{\theta}\right]$ such that $Q_k^H(\theta^*, b')$ converges to some positive number along some subsequence,

$$d^H \liminf_{k\to\infty} Q_k^H \left(\theta^*, b'\right) = \varepsilon > 0. \tag{29}$$

We prove that this implies $b' \geq v - \bar{V}$. This implies the claim.

Let $\theta_k \equiv \sup \{\theta | \beta_k (\theta) \leq b'\}$. We prove that (29) implies

$$\liminf_{k\to\infty} (1 - \xi_k^H (\theta_k)) D_k^H \geq \varepsilon. \tag{30}$$

We are done if $\liminf_{k\to\infty} \xi_k^H (\theta_k) < 1$. So, suppose that $\xi_k^H (\theta_k) \to 1$. The inequality now follows from the following chain of equations. For k sufficiently large,

$$\begin{aligned}
\varepsilon &\leq d^H (1 - \xi_k^H (\theta_k)) + \delta_k d^H \left(1 - \xi_k^H (\theta_k)\right) \xi_k^H \left(\theta_k^1 (\theta)\right) + \delta_k^2 \left(1 - \xi_k^H (\theta_k)\right) .... \\
&= d^H (1 - \xi_k^H (\theta_k)) + \delta_k d^H \left(1 - \xi_k^H (\theta_k)\right) \left[\xi_k^H \left(\theta_k^1 (\theta)\right) + \delta_k \xi_k^H \left(\theta_k^1 (\theta)\right) \xi_k^H \left(\theta_k^2 (\theta)\right) + ...\right] \\
&\leq d^H (1 - \xi_k^H (\theta_k)) + \delta_k d^H \left(1 - \xi_k^H (\theta_k)\right) \left[\xi_k^H \left(\bar{\theta}\right) + \delta_k \xi_k^H \left(\theta_k^1 \left(\bar{\theta}\right)\right) \xi_k^H \left(\bar{\theta}\right) + ...\right],
\end{aligned}$$

where the first inequality comes from the definition of Q_k^H and the second inequality comes from $\theta_k^1 (\theta) \to 1 > \overline{\theta}$ and ξ_k^H being nonincreasing. Integrating both sides with respect to G^H, taking limits with $k \to \infty$, and noting that $\xi_k^H (\theta_k) \to 1$, we rewrite further

$$\begin{aligned}
&\liminf_{k\to\infty} \left(1 - \xi_k^H (\theta_k)\right) d^H \int_0^1 \left[\xi_k^H \left(\bar{\theta}\right) + \delta_k \xi_k^H \left(\theta_k^1 \left(\bar{\theta}\right)\right) \xi_k^H \left(\bar{\theta}\right) + ...\right] dG^H \\
\leq\ &\liminf_{k\to\infty} \left(1 - \xi_k^H (\theta_k)\right) d^H \int_0^1 \left[1 + \delta_k \xi_k^H (\tau) + ...\right] dG^H (\tau) \\
=\ &\lim \left(1 - \xi_k^H (\theta_k)\right) D_k^H.
\end{aligned}$$

where we used that ξ_k^H is nonincreasing, $\xi_k^H \left(\bar{\theta}\right) \to 1$, and the steady-state conditions. Together, the two displayed chains of equations imply the desired inequality (30).

We expand (30) using the definition of ξ_k^w,

$$\liminf_{k\to\infty} D_k^H e^{-D_k^H (1-\Gamma_k^H (\theta_k))/S_k^H} \geq \varepsilon. \tag{31}$$

Equation (31) implies that $\lim_{k\to\infty} \Gamma_k^H (\theta_k) = 1$: Otherwise, if $\lim sup \Gamma_k^H (\theta_k) < 1$ were true, $S_k^H \to 1$ and $D_k^H \to \infty$ would imply that $\liminf_{k\to\infty} D_k^H e^{-D_k^H (1-\Gamma_k^H (\theta_k))/S_k^H} = 0$ by l'Hospital's rule,[33] contradicting (31). Using $D_k^L \to d^L < 1$, we obtain that

$$\limsup D_k^L \Gamma_k^L(\theta_k) e^{-D_k^L(1-\Gamma_k^L(\theta_k)/S_k^L} \leq 1.$$

Hence,

$$\liminf_{k\to\infty} \frac{D_k^H \Gamma_k^H(\theta_k) e^{-D_k^H(1-\Gamma_k^H(\theta_k)/S_k^H}}{D_k^L \Gamma_k^L(\theta_k) e^{-D_k^L(1-\Gamma_k^L(\theta_k)/S_k^L}} \geq \varepsilon. \tag{32}$$

The likelihood ratio of tying satisfies

$$\begin{aligned}
\frac{\theta_k^0(\theta_k)}{1-\theta_k^0(\theta_k)} &= \frac{\theta_k}{1-\theta_k} \frac{\gamma_k^H(\theta_k)}{\gamma_k^L(\theta_k)} \frac{D_k^H}{D_k^L} \frac{S_k^L}{S_k^H} \frac{e^{-D_k^H(1-\Gamma_k^H(\theta_k)/S_k^H}}{e^{-D_k^L(1-\Gamma_k^L(\theta_k))/S_k^L}} \\
&\geq \frac{\theta_k}{1-\theta_k} \frac{\Gamma_k^H(\theta_k)}{\Gamma_k^L(\theta_k)} \frac{D_k^H}{D_k^L} \frac{S_k^L}{S_k^H} \frac{e^{-D_k^H(1-\Gamma_k^H(\theta_k))/S_k^H}}{e^{-D_k^L(1-\Gamma_k^L(\theta_k))/S_k^L}}.
\end{aligned}$$

The inequality follows from the MLRP of Γ^w, $\frac{\gamma_k^H(\theta_k)}{\gamma_k^L(\theta_k)} \geq \frac{\Gamma_k^H(\theta_k)}{\Gamma_k^L(\theta_k)}$. Using (32) (for the first inequality) and using $\frac{S_k^L}{S_k^H} \to \infty$ (for the second equality),

$$\liminf_{k\to\infty} \frac{\theta_k^0(\theta_k)}{1-\theta_k^0(\theta_k)} \geq \varepsilon \liminf_{k\to\infty} \frac{\theta_k}{1-\theta_k} \frac{S_k^L}{S_k^H} = \infty. \tag{33}$$

Therefore, the posterior $\theta_k^0(\theta_k, \theta_k) \to 1$.

We show that $\theta_k^0(\theta_k, \theta_k) \to 1$ implies that $\beta_k(\theta_k) \to v - \bar{V}$; that is, $b' = v - \bar{V}$, as claimed in the beginning. From Lemma 18, $\beta_k(\theta_k) = v - \delta_k EU\left(\theta_k^+(\theta_k, \theta_k), \beta_k | \theta_k^0(\theta_k, \theta_k)\right)$. By $\theta_k^+(\theta_k, \theta_k) \to 1$ and because the sequence of payoffs $EU\left(\theta, \beta_k | \theta'\right)$ is Lipschitz continuous in θ' with a uniform Lipschitz constant, we can pass the limit through; that is $\lim EU\left(\theta_k^+(\theta_k, \theta_k), \beta_k | H\right) = \lim EU(1, \beta_k | H) = \bar{V}$. Hence,

$$\begin{aligned}
& \lim \beta_k(\theta_k) \\
=\ & v - \lim \delta_k EU\left(\theta_k^+(\theta_k, \theta_k), \beta_k | \theta_k^0(\theta_k, \theta_k)\right) \\
=\ & v - \lim \underbrace{\theta_k^0(\theta_k, \theta_k)}_{\to 1} \underbrace{EU\left(\theta_k^+(\theta_k, \theta_k), \beta_k | H\right)}_{\to \bar{V}} - \underbrace{\left(1-\theta_k^0(\theta_k, \theta_k)\right)}_{\to 0} EU\left(\theta_k^+(\theta_k, \theta_k), \beta_k | L\right) \\
=\ & v - \bar{V} = b'.
\end{aligned}$$

Thus, $\lim Q_k^H(\theta, p) = 0$ for all $p < v - \bar{V}$, $\theta \in [\underline{\theta}, \overline{\theta}]$: If not, then $\liminf Q_k^H\left(\theta', p'\right) > 0$ for some $p' < v - \bar{V}$ and θ'. As we have shown before, $\liminf Q_k^H\left(\theta', p'\right) > 0$ implies $p' = v - \bar{V}$, a contradiction.

From $\beta_k(\theta) \leq \beta_k(1) = v - \delta V_k(1)$ for all θ and k, it follows that $\lim Q_k^H(\theta, p) =$

[33] Let $x_k = D_k^H$ and $c_k = (1-\Gamma_k^H(\theta_k))/S_k^H$. Using l'Hospital's rule, $\liminf x_k e^{-x_k c_k} = 0$ if $x_k \to \infty$ and $\liminf c_k > 0$. Hence, $c_k = (1-\Gamma_k^H(\theta_k))/S_k^H$ must converge to 0. From before, $S_k^H \to 1$, so this requires $\Gamma_k^H(\theta_k) \to 1$.

$1/d^H$ for all $p > v - \bar{V}$, $\theta \in [\underline{\theta}, \overline{\theta}]$. Hence,

$$\lim_{k\to\infty} E_k [p, \theta | H] = \lim \int_0^1 \left(1 - \frac{Q_k^H (\theta, b)}{Q_k^H (\theta)}\right) db = v - \bar{V}.$$

□

Proof of Proposition 2.

We have shown that the trading probabilities become competitive in Lemma 10. We now show that prices become competitive, too.

Case 1: $d^L < d^H < 1$. By Lemma 1, in both states, the probability of being the sole bidder becomes one, $e^{-D_k^w/S_k^w} \to 1$, which implies $V_k(\theta) \to v$ for all θ by inspection of the payoffs. In particular, $V_k(1) \to v$. Hence, the bidding strategy $\beta_k(\theta) \to 0$ for all θ. Because the expected price is smaller than $\beta_k(1)$ by definition and the monotonicity of β_k and because $\beta_k(1) \to 0$, $\lim_{k\to\infty} P_k^w(\theta) = \lim_{k\to\infty} P_k^w(S) = 0$ follows.

Case 2: $d^L > 1$ and $d^H > 1$. We show $EU_k[\beta_k, \theta = 0|L] \to 0$. By monotonicity of the bidding strategy, $\beta_k(0) \le \beta_k(\theta)$ for all θ; hence, a buyer with belief $\theta = 0$ wins only as the sole bidder,

$$\begin{aligned} EU_k[\beta_k, \theta = 0|L] &= e^{-D_k^L/S_k^L}(v) + \delta_k \left(1 - e^{-D_k^L/S_k^L}\right) EU_k[\beta_k, \theta = 0|L] \\ \Leftrightarrow EU_k[\beta_k, \theta = 0|L] &= \frac{e^{-D_k^L/S_k^L}}{\frac{1-\delta_k}{e^{-D_k^L/S_k^L}} + \delta_k} v. \end{aligned}$$

From Lemma 1, $\frac{1-\delta_k}{e^{-D_k^L/S_k^L}} \to \infty$, while $e^{-D_k^L/S_k^L} \to 0$. Therefore,

$$\lim_{k\to\infty} EU_k[\beta_k, \theta = 0|L] = 0.$$

Since $EU_k[\beta_k, \theta = 0|L] = V_k(0)$ and $V_k(0) \to 0$, we have $\lim \beta_k(\theta) = v$ for all θ. Because a bidder is never a sole bidder in the limit, $\lim_{k\to\infty} P_k^w(\theta) = \lim_{k\to\infty} P_k^w(S) = v$ follows.

Case 3: $d^H > 1$ and $d^L < 1$.

As in Case 1, by Lemma 1, $\lim_{k\to\infty} P_k^L(\theta) = \lim_{k\to\infty} P_k^L(S) = 0$. We now argue $w = H$. From before, $\theta_k^1(\underline{\theta}) \to 1$. Since, in the high state, the first cohort has a vanishing winning probability, and since there is no exogenous exit in the limit either, it follows that

$$\lim_{k\to\infty} V_k^H(\underline{\theta}) = \lim_{k\to\infty} V_k^H(\theta_k^1(\underline{\theta})) = \bar{V}.$$

From Lemma 9, $Q_k^H(\underline{\theta}) \to 1/d^H$. From Lemma 11 $P_k^H(\underline{\theta}) = v - \bar{V}$. Together,

$$\lim_{k\to\infty} V_k^H(\underline{\theta}) = \frac{v-(v-\bar{V})}{d^H} = \frac{\bar{V}}{d^H}.$$

Since $d^H > 1$, $\bar{V} = \frac{\bar{V}}{d^H}$ implies that $\bar{V} = 0$. Thus, the expected price $\lim_{k\to\infty} P_k^H(\theta) = \lim_{k\to\infty} P_k^H(S) = v - \bar{V} = v$. □

8.5 Proof of Proposition 3: Time Pattern of Bids

Proof. Step 1: We derive some auxiliary observations. By definition, $\theta_k^t(\theta)$ satisfies

$$\frac{\theta_k^t(\theta)}{1-\theta_k^t(\theta)} = \frac{\theta_k^{t-1}(\theta)}{1-\theta_k^{t-1}(\theta)} \frac{1-\Gamma_{(1),k}^H(\theta_k^{t-1}(\theta))}{1-\Gamma_{(1),k}^L(\theta_k^{t-1}(\theta))}.$$

The belief $\theta_k^t(\theta)$ is strictly increasing in t. From the proof of Proposition 2, $\lim_{k\to\infty} \Gamma_{(1),k}^H(\theta) = \lim_{k\to\infty} 1-\Gamma_{(1),k}^L(\theta) = 0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. Therefore, $\lim_{k\to\infty} \theta_k^1(\theta) = 1$; thus, by monotonicity, $\lim_{k\to\infty} \theta_k^t(\theta) = 1$. The posterior after losing t times and then being tied satisfies (if densities are positive)

$$\frac{\theta_k^0\left(\theta_k^t(\theta), \theta_k^t(\theta)\right)}{1-\theta_k^0\left(\theta_k^t(\theta), \theta_k^t(\theta)\right)} = \frac{\theta_k^t(\theta)}{1-\theta_k^t(\theta)} \frac{\gamma_{(1),k}^H(\theta_k^t(\theta))}{\gamma_{(1),k}^L(\theta_k^t(\theta))}. \tag{34}$$

By definition, $\gamma_{(1),k}^w = \mu_k^w \gamma_k^w \Gamma_{(1),k}^w$. Using the no-introspection condition to substitute for γ_k^w,

$$\frac{\gamma_{(1),k}^H(\theta)}{\gamma_{(1),k}^L(\theta)} = \frac{\theta}{1-\theta} \frac{S_k^L}{S_k^H} \frac{\Gamma_{(1),k}^H(\theta)}{\Gamma_{(1),k}^L(\theta)}. \tag{35}$$

Substituting iteratively into (34),

$$\begin{aligned}
\frac{\theta_k^0\left(\theta_k^t(\theta), \theta_k^t(\theta)\right)}{1-\theta_k^0\left(\theta_k^t(\theta), \theta_k^t(\theta)\right)} &= \frac{\theta_k^{t-1}}{1-\theta_k^{t-1}} \frac{1-\Gamma_{(1),k}^H(\theta_k^{t-1})}{1-\Gamma_{(1),k}^L(\theta_k^{t-1})} \frac{\theta_k^t}{1-\theta_k^t} \frac{S_k^L}{S_k^H} \frac{\Gamma_{(1),k}^H(\theta_k^t)}{\Gamma_{(1),k}^L(\theta_k^t)} = \\
&= \left(\frac{\theta_k^{t-1}}{1-\theta_k^{t-1}} \frac{1-\Gamma_{(1),k}^H(\theta_k^{t-1})}{1-\Gamma_{(1),k}^L(\theta_k^{t-1})}\right)^2 \frac{S_k^L}{S_k^H} \frac{\Gamma_{(1),k}^H(\theta_k^t)}{\Gamma_{(1),k}^L(\theta_k^t)} = \\
&= \left(\frac{\theta_k^{t-2}}{1-\theta_k^{t-2}} \frac{1-\Gamma_{(1),k}^H(\theta_k^{t-2})}{1-\Gamma_{(1),k}^L(\theta_k^{t-2})} \frac{1-\Gamma_{(1),k}^H(\theta_k^{t-1})}{1-\Gamma_{(1),k}^L(\theta_k^{t-1})}\right)^2 \frac{S_k^L}{S_k^H} \frac{\Gamma_{(1),k}^H(\theta_k^t)}{\Gamma_{(1),k}^L(\theta_k^t)} \\
&= \left(\frac{\theta}{1-\theta} \frac{1-\Gamma_{(1),k}^H(\theta)}{1-\Gamma_{(1),k}^L(\theta)} \cdots \frac{1-\Gamma_{(1),k}^H(\theta_k^{t-1})}{1-\Gamma_{(1),k}^L(\theta_k^{t-1})}\right)^2 \frac{S_k^L}{S_k^H} \frac{\Gamma_{(1),k}^H(\theta_k^t)}{\Gamma_{(1),k}^L(\theta_k^t)}
\end{aligned} \tag{36}$$

Recall from the proof of Proposition 2 that

$$\lim_{k\to\infty}\mu_k^L = 0 \text{ and } \lim_{k\to\infty}\frac{\mu_k^L}{1-\delta_k} = \frac{d^L}{1-d^L}. \tag{37}$$

L'Hospital's rule implies that $\lim_{x\to 0}\frac{1-e^{-x}}{x} = 1$. Hence,

$$\lim_{k\to\infty}\frac{1-e^{-\mu_k^L}}{1-\delta_k} = \frac{d^L}{1-d^L}. \tag{38}$$

Step 2: Proof of Statement (i) of Proposition 3.

From the previous step, $\lim \theta_k^1(\theta) > \overline{\theta}$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. Thus, monotonicity of beliefs implies that for sufficiently large k,

$$\theta_k^0\left(\theta_k^t(\underline{\theta}), \theta_k^t(\underline{\theta})\right) \geq \theta_k^0\left(\theta_k^{t-1}(\theta), \theta_k^{t-1}(\theta)\right)$$

for all $\theta \in [\underline{\theta}, \overline{\theta}]$.[34] Therefore, it is sufficient to prove statement (i) at $\theta = \underline{\theta}$. In the following, we simplify

$$\theta_k^t \equiv \theta_k^t\left(\underline{\theta}\right).$$

For sufficiently large k such that $\theta_k^1 > \overline{\theta}$, the steady-state conditions imply

$$D_k^w\Gamma_k^w(\theta_k^t) = d^w + \int_{\underline{\theta}}^{\bar{\theta}}\delta_k\xi_k^w(\tau)\,dG^w(\tau) + \ldots + \int_{\underline{\theta}}^{\bar{\theta}}\delta_k^{t-1}\xi_k^w(\tau)\ldots\xi_k^w\left(\theta_k^{t-1}(\tau)\right)dG^w(\tau) \leq td^w.$$

From the proof of Proposition 2, $D_k^H \to \infty$. Hence,

$$\lim_{k\to\infty}\frac{\Gamma_{(1),k}^H(\theta_k^t)}{e^{-\mu_k^H}} = \lim_{k\to\infty}\frac{e^{-\mu_k^H\left(1-\frac{td^H}{D_k^H}\right)}}{e^{-\mu_k^H}} = 1. \tag{39}$$

Moreover, the steady-state conditions imply that

$$D_k^w\left(1-\Gamma_k^w(\theta_k^t)\right) \leq \frac{d^L\left(1-e^{-\mu_k^L}\right)^t}{1-\delta_k+e^{-\mu_k^L}} \tag{40}$$

[34] The tying posterior θ_k^0 is defined at $\underline{\theta}$ by definition of γ_k^w and $g_k^w(\underline{\theta}) > 0$, which ensure $\gamma_{(1)}^w\left(\theta_k^t(\underline{\theta})\right) > 0$ for all t.

because $\xi_k^w(\theta) \leq \xi_k^w(\underline{\theta}) = 1 - e^{-\mu_k^L}$ for all $\theta \geq \underline{\theta}$ and

$$\begin{aligned} D_k^w\left(1 - \Gamma_k^w(\theta_k^t)\right) &= \int_{\underline{\theta}}^{\bar{\theta}} \delta_k^t \xi_k^w(\tau) \ldots \xi_k^w\left(\theta_k^t(\tau)\right) dG^w(\tau) \\ &\quad + \int_{\underline{\theta}}^{\bar{\theta}} \delta_k^{t+1} \xi_k^w(\tau) \ldots \xi_k^w\left(\theta_k^{t+1}(\tau)\right) dG^w(\tau) + \ldots. \\ &\leq \delta_k^t \left(1 - e^{-\mu_k^L}\right)^t d^L + \delta_k^{t+1}\left(1 - e^{-\mu_k^L}\right)^{t+1} d^L + \ldots. \\ &= \frac{1}{1 - \delta_k + e^{-\mu_k^L}} \delta_k^t \left(1 - e^{-\mu_k^L}\right)^t d^L. \end{aligned}$$

From (38) and (39),

$$\lim_{k\to\infty} \frac{1 - \Gamma_{(1),k}^L(\underline{\theta})}{1 - \delta_k} = \lim_{k\to\infty} \frac{1 - e^{-\mu_k^L}}{1 - \delta_k} = \frac{d^L}{1 - d^L}.$$

Hence, (40) implies

$$\limsup_{k\to\infty} \frac{D_k^L\left(1 - \Gamma_k^L(\theta_k^t)\right)}{(1 - \delta_k)^t} \leq d^L \left(\frac{d^L}{1 - d^L}\right)^t.$$

This implies

$$\limsup_{k\to\infty} \frac{1 - \Gamma_{(1),k}^L(\theta_k^t)}{(1 - \delta_k)^{t+1}} = \limsup_{k\to\infty} \frac{1 - e^{-\mu_k^L\left(1 - \Gamma_k^L(\theta_k^t)\right)}}{(1 - \delta_k)^{t+1}} \leq \left(\frac{d^L}{1 - d^L}\right)^{t+1}, \tag{41}$$

since,

$$\limsup_{k\to\infty} \frac{\mu_k^L\left(1 - \Gamma_k^L(\theta_k^t)\right)}{(1 - \delta_k)^{t+1}} \leq \limsup_{k\to\infty} \frac{1}{(1 - \delta_k) S_k^L} \left(\frac{D_k^L\left(1 - \Gamma_k^L(\theta_k^t)\right)}{(1 - \delta_k)^t}\right) = \frac{d^L}{1 - d^L}\left(\frac{d^L}{1 - d^L}\right)^t.$$

Note that (20), (23), and (39) imply

$$\lim S_k^L \Gamma_{(1),k}^H(\theta_k^t) = \lim \frac{1 - d^L}{1 - \delta_k} e^{-\frac{1 - d^H}{1 - \delta_k}}. \tag{42}$$

Taking limits on (36) and ignoring all terms that are finite and non-zero,

$$\begin{aligned} &\lim_{k\to\infty} \frac{\theta_k^0\left(\theta_k^t(\bar{\theta}), \theta_k^t(\bar{\theta})\right)}{1 - \theta_k^0\left(\theta_k^t(\bar{\theta}), \theta_k^t(\bar{\theta})\right)} \\ = &\lim_{k\to\infty} \left(\frac{1}{1 - \Gamma_{(1),k}^L(\underline{\theta})} \cdots \frac{1}{1 - \Gamma_{(1),k}^L(\theta_k^{t-2})} \frac{1}{1 - \Gamma_{(1),k}^L(\theta_k^{t-1})}\right)^2 S_k^L \Gamma_{(1),k}^H(\theta_k^t) \\ = &\lim_{k\to\infty} \frac{\frac{1}{1-\delta_k} e^{-\left(1 - d^H\right)/(1 - \delta_k)}}{(1 - \delta_k)(1 - \delta_k)^2 \ldots (1 - \delta_k)^t} = 0, \end{aligned}$$

where we used (42) and (41) for the second and l'Hospital's rule for the final equality.

From the proof of Proposition 2, $\lim EU\left(\theta, \beta_k | 0\right) = v$ for all θ. Because the sequence of payoffs $EU\left(\theta, \beta_k | \theta'\right)$ is Lipschitz continuous in θ' with a uniform Lipschitz constant, we can pass the limit through:

$$\begin{aligned}
\lim_{k\to\infty} \beta_k(\theta_k^t) &= v - \lim_{k\to\infty} \delta_k EU\left(\theta_k^+\left(\theta_k^t, \theta_k^t\right), \beta_k | \theta_k^0\left(\theta_k^t, \theta_k^t\right)\right) \\
&= v - \lim_{k\to\infty} EU\left(1, \beta_k | 0\right) = 0.
\end{aligned}$$

This concludes the proof of statement (i).

Step 3: Proof of Statement (ii) of Proposition 3.

Let $t_k \equiv \frac{-(0.5)}{(1-\delta)\ln(1-\delta)}$. Note that Lemma 3 implies that the likelihood ratio of losing, $\frac{1-\Gamma^H_{(1),k}(\theta)}{1-\Gamma^L_{(1),k}(\theta)}$, is a nondecreasing function of θ on the support of $\Gamma^w_{(1),k}$. Therefore, (36) implies for $\theta \in \left[\underline{\theta}, \bar{\theta}\right]$:

$$\frac{\theta_k^0\left(\theta_k^t\left(\theta\right), \theta_k^t\left(\theta\right)\right)}{1-\theta_k^0\left(\theta_k^t\left(\theta\right), \theta_k^t\left(\theta\right)\right)} \geq \left(\frac{\theta}{1-\theta}\right)^2 \left(\frac{1-\Gamma^H_{(1),k}(\theta)}{1-\Gamma^L_{(1),k}(\theta)}\right)^{2t_k} \frac{S_k^L}{S_k^H} \frac{\Gamma^H_{(1),k}(\theta_k^t)}{\Gamma^L_{(1),k}(\theta_k^t)}.$$

Evaluating the limit of terms on the right-hand side:

$$\begin{aligned}
&\liminf_{k\to\infty} \left(\frac{1-\Gamma^H_{(1),k}(\theta)}{1-\Gamma^L_{(1),k}(\theta)}\right)^{2t_k} S_k^L \Gamma^H_{(1),k}(\theta_k^t) \\
= &\liminf_{k\to\infty} \left(\frac{1}{1-\Gamma^L_{(1),k}(\theta)}\right)^{2t_k} \frac{1-d^L}{1-\delta_k} e^{-\frac{1-d^H}{1-\delta_k}} \\
= &\liminf_{k\to\infty} \left(\frac{d^L}{1-d^L}\left(1-\delta_k\right)\right)^{-2t_k} \frac{1-d^L}{1-\delta_k} e^{-\frac{1-d^H}{1-\delta_k}} = \infty,
\end{aligned}$$

where we used $\Gamma^H_{(1),k}(\theta) \to 0$ and (42) for the second line, (38) for the first equality of the third line, and the following observation for the second equality of the third line:

$$\lim_{(1-\delta)\to 0} \frac{\frac{1-d^L}{1-\delta} e^{-\frac{1-d^H}{1-\delta}}}{\left(\frac{d_L}{1-d_L}\left(1-\delta\right)\right)^{-\frac{1}{(1-\delta)\ln(1-\delta)}}} = \lim_{(1-\delta)\to 0} \frac{1-d^L}{1-\delta} \left(\left(\frac{d_L}{1-d_L}\right)^{\frac{1}{\ln(1-\delta)}} \frac{e^{1-d_H}}{e} \right)^{\frac{-1}{1-\delta}} = \infty.$$

This follows from $\lim\left(\frac{d_L}{1-d_L}\right)^{\frac{1}{\ln(1-\delta)}} = 1$ and $\lim\left(\frac{d_L}{1-d_L}\right)^{\frac{1}{\ln(1-\delta)}} \frac{e^{1-d_H}}{e} = e^{-d_H} < 1$, so that $\lim_{(1-\delta)\to 0} \left(\left(\frac{d_L}{1-d_L}\right)^{\frac{1}{\ln(1-\delta)}} \frac{e^{1-d_H}}{e}\right)^{\frac{-1}{1-\delta}} = \infty$.

Thus, $\lim_{k\to\infty} \theta_k^0\left(\theta_k^t\left(\theta\right), \theta_k^t\left(\theta\right)\right) = 1$, as claimed. From the proof of Proposition 2,

$\lim EU\left(1,\beta_k|1\right)=0$. Together,

$$\begin{aligned}\lim_{k\to\infty}\beta_k(\theta_k^{t_k}) &= v-\lim_{k\to\infty}\delta_k EU\left(\theta_k^{+}\left(\theta_k^{t_k},\theta_k^{t_k}\right),\beta_k|\theta_k^{0}\left(\theta_k^{t_k},\theta_k^{t_k}\right)\right)\\ &= v-\lim_{k\to\infty}\delta_k EU\left(1,\beta_k|1\right)=v.\end{aligned}$$

□

9 Heterogeneous Buyers and Strategic Sellers

9.1 The Setup

We start by developing some helpful notation. Let $q_S^w\left(\theta,\rho\right)$ be the trading probability of a seller with belief θ in the current period who follows strategy ρ, conditional on state w. Similarly, $Q_S^w\left(\theta,\rho\right)$ and $P_S^w\left(\theta,\rho\right)$ are the lifetime trading probability and the expected price conditional on trading some time during the lifetime of a seller who follows strategy ρ with belief θ if the state is w. $E\Pi\left[\left(\theta,\rho\right)|w\right]=Q_S^w\left(\theta,\rho\right)P_S^w\left(\theta,\rho\right)$ is the expected profit. We extend the definition of $q_S^w\left(\theta,\rho\right)$ to

$$q_S^{\theta}\left(\theta',\rho\right)=\theta q_S^{H}\left(\theta',\rho\right)+\left(1-\theta\right)q_S^{L}\left(\theta',\rho\right),$$

and define $E\Pi\left[\left(\theta',\rho\right)|\theta\right]$, Q_S^{θ}, and P_S^{θ} similarly. Equilibrium payoffs are $V_S\left(\theta\right)=E\Pi\left[\left(\theta,\rho\right)|\theta\right]$. For buyers, we define the current trading probability $q_B^w\left(\theta,v,\beta\right)$, the expected payoff $EU\left[\theta',v,\beta|w\right]$, and equilibrium payoffs $V_B\left(\theta,v\right)$. These definitions are extended to $EU\left[\theta',v,\beta|\theta\right]$, Q_B^{θ} and P_B^{θ}.

We now state the equilibrium requirements formally.

Sequential Rationality.— A (Markovian and pure) bidding strategy $\beta:[0,1]^2\to[0,1]$ is sequentially rational if

$$\beta\left(\theta,v,r\right)\in\arg\max_b q_B^{\theta}\left(b,r\right)\left(v-p_B^{\theta}\left(b,r\right)\right)+\left(1-q_B^{\theta}\left(b,r\right)\right)\delta EU\left[\left(\theta_B^{+}\left(\theta,r,b\right),v,\beta\right)|\theta_B^{+}\left(\theta,r,b\right)\right],$$

for all (θ,v,r), where $q_B^{\theta}\left(b,r\right)$ and $p_B^{\theta}\left(b,r\right)$ are the expected trading probability with bid b and the expected price, respectively. A (Markovian and pure) reserve price strategy $\rho:[0,1]\to[0,1]$ is sequentially rational if, for all θ,

$$\rho\left(\theta\right)\in\arg\max_r q_S^{\theta}\left(r\right)p_S^{\theta}\left(r\right)+\left(1-q_S^{\theta}\left(r\right)\right)\delta E\Pi\left[\left(\theta_S^{+}\left(\theta,r\right),\rho\right)|\theta_S^{+}\left(\theta,r\right)\right],$$

where $q_S^{\theta}\left(r\right)$ and $p_S^{\theta}\left(r\right)$ are the expected trading probability and price given r. Recall that a strategy profile is *monotone* if β and ρ are weakly increasing in all their arguments.

Steady-State Conditions.—First, for all sets $A\subset[0,1]$, the mass of the sellers with types in A at the beginning of a period equals the mass of sellers with types in A at the end of a period,

$$S^w \Gamma_S^w (A) = G_S^w (A) + S^w \delta \int_{\left\{\theta : \theta_S^+ (\theta, \rho(\theta)) \in A\right\}} \left(1 - q_S^w \left(\theta, \rho\left(\theta\right)\right)\right) d\Gamma_S^w.$$

Second, for buyers, for every $X \subset [0,1]^2$, the mass $D^w \Gamma_B^w (X)$ is equal to

$$d^w G_B^w (X) + D^w \delta \int \left(\left(\Gamma_S^w \left(X^I (\theta, v)\right)\right) + \int_{\theta' \in X^+(\theta, v)} \left(1 - q_B^w \left(\beta\left(\theta, v, \rho\left(\theta'\right)\right), \rho\left(\theta'\right)\right)\right) d\Gamma_S^w \right) d\Gamma_B^w$$

where $X^I (\theta, v)$ is the set of sellers' beliefs such that a buyer of type (θ, v) does not participate given the sellers' reserve price and updates to a type in X, and $X^+ (\theta, v)$ is the set of sellers' beliefs such that a buyer of type (θ, v) participates and updates to a type in X after losing,[35] and $q_B^w \left(\beta\left(\theta, v, \rho\left(\theta'\right)\right), \rho\left(\theta'\right)\right)$ is the probability to win with a bid $\beta\left(\theta, v, \rho\left(\theta'\right)\right)$ if the reserve price is $\rho\left(\theta'\right)$. As for the base model, we take the steady-state conditions above as fundamentals; see the footnote on Page 11.

Bayesian Updating.—Suppose that the bidding strategy and the reserve price strategy are strictly increasing in all their arguments. If $r = \rho\left(\theta'\right)$, then Bayes' formula and no-introspection require that the interim belief satisfies[36]

$$\frac{\theta^I (\theta, r)}{1 - \theta^I (\theta, r)} = \frac{\theta}{1 - \theta} \frac{S^H \gamma_S^H \left(\theta'\right)}{S^L \gamma_S^L \left(\theta'\right)} = \frac{\theta}{1 - \theta} \frac{\theta'}{1 - \theta'}. \tag{43}$$

Let $A(r, b) = \{(\theta, v) : \beta(\theta, v, r) \geq b\}$, the set of buyers bidding above b. Then,

$$e^{-\mu^w \Gamma_B^w (A_k(r,b)))}$$

is the probability that the seller is matched with *no* buyer having a type in $A(r, b)$; recall (1). A buyer's posterior conditional on losing with a bid $b \geq r$ is[37]

$$\theta_B^+ (\theta, r, b) = \frac{\theta(1 - e^{-\mu_k^H \Gamma_B^H (A(r,b))})}{\theta(1 - e^{-\mu_k^H \Gamma_B^H (A(r,b))}) + (1 - \theta)\left(1 - e^{-\mu^L \Gamma_B^L (A(r,b))}\right)}.$$

A seller's posterior conditional on no participation at a reserve price r is

$$\theta_S^+ (\theta, r) = \frac{\theta e^{-\mu^H (1 - \Gamma_B^H (A(r,r))}}{\theta e^{-\mu^H \Gamma_B^H (A(r,r))} + (1 - \theta) e^{-\mu^L \Gamma_B^L (A_k(r,r))}}.$$

[35] Thus, $X^I (\theta, v) = \left\{\theta' | \beta(\theta, v, \rho(\theta')) < \rho(\theta') \text{ and } \theta^I (\theta, v) \in X\right\}$ and $X^+ (\theta, v) = \left\{\theta' | \beta(\theta, v, \rho(\theta')) \geq \rho(\theta') \text{ and } \theta^+ \left(\theta^I, \rho(\theta'), \beta\left(\left(\theta^I, v\right)\right)\right) \in X\right\}$, with $\theta^I = \theta^I (\theta, \rho(\theta'))$.

[36] If θ' is not in the support of the distribution of sellers' beliefs, then the second equality in (43) is a definitional extension of Bayesian updating.

[37] We extend the posterior as in the base model; see the remarks following the statement of Bayes' rule in equation (2).

The above requirements extend to strategy profiles that are weakly increasing, by taking the possibility of ties appropriately into account. Finally, recall that beliefs are *passive* if $\theta^I(\theta, r) = \theta$ whenever $r \neq \rho(\theta')$ for all θ'.

Equilibrium Definition.—A list $(\beta, \rho, S^w, D^w, \Gamma_S^w, \Gamma_B^w, \theta_S^+, \theta_B^+, \theta^I)$ is a *monotone steady-state equilibrium with passive beliefs* if the stock $(S^w, D^w, \Gamma_S^w, \Gamma_B^w)$ satisfies the steady-state conditions, if strategies β and ρ are sequentially rational and monotone, and if beliefs θ_S^+, θ_B^+, and θ^I are consistent with Bayesian updating and off-equilibrium beliefs are passive.

9.2 Steady-State Implications and Auxiliary Results

Take a sequence of survival rates $\{\delta_k\}_{k=1}^{\infty}$ with

$$\lim_{k\to\infty}(1-\delta_k) = 0.$$

Suppose a sequence of equilibria exist. Denote the equilibrium magnitudes corresponding to δ_k by subscripts k, such as β_k, $\rho_k, \Gamma_{S,k}^w$, θ_k^I, etc. Each equilibrium gives rise to an outcome. As in the benchmark model, let $Q_{B,k}^w(\theta, v)$, $Q_{S,k}^w(\theta)$ and $P_{B,k}^w(\theta, v)$, $P_{S,k}^w(\theta)$ be the equilibrium trading probability and price, respectively. A sequence of outcomes $(Q_{B,k}^w, Q_{S,k}^w, P_{B,k}^w, P_{S,k}^w)$ converges to the competitive outcome if it converges pointwise everywhere, except possibly for buyer types who have the marginal valuation $v = p_*^w$ and types who are convinced of the "wrong" state ($\theta = 1$ for $w = L$ and $\theta = 0$ for $w = H$).

Steady-State Implications.—Steady state requires that the mass of buyers and sellers who enter and expect to trade is equal, so that

$$d^w \int_{[0,1]^2} Q_{B,k}^w(\theta, v)\, dG_B^w = \int_{[0,1]} Q_{S,k}^w(\theta)\, dG_S^w. \tag{44}$$

Similarly, the total transfer made by buyers is equal to the total transfers received by sellers, so that

$$d^w \int_{[0,1]^2} P_{B,k}^w(\theta, v)\, Q_{B,k}^w(\theta, v)\, dG_B^w = \int_{[0,1]} P_{S,k}^w(\theta)\, Q_{S,k}^w(\theta)\, dG_S^w. \tag{45}$$

An outcome $\left(Q_{B,k}^w, Q_{S,k}^w, P_{B,k}^w, P_{S,k}^w\right)$ that satisfies the above conditions is *feasible.* A feasible outcome corresponds to an allocation in the quasilinear economy defined by the population of entering buyers and sellers.

In a steady-state, given any set $Z_k \subset [0,1]$, the mass of buyers having valuations $v \in Z_k$ who trade in any given period equals the mass of buyers having valuations

$v \in Z_k$ who enter the market, that is,

$$D_k^w \int_{[0,1]\times Z_k} q_{B,k}^w (\theta, v)\, d\Gamma_{B,k}^w = d^w \int_{[0,1]\times Z_k} Q_{B,k}^w (\theta, v)\, dG_B^w. \tag{46}$$

Similarly, the mass of buyers who exit without trading equals the mass of buyers who enter and expect to exit without trading,

$$D_k^w \int_{[0,1]\times Z_k} \left(1 - q_{B,k}^w (\theta, v)\right) (1-\delta_k)\, d\Gamma_{B,k}^w = d^w \int_{[0,1]\times Z_k} \left(1 - Q_{B,k}^w (\theta, v)\right) dG_B^w. \tag{47}$$

Proving that the above are indeed implications of the steady-state conditions is tedious and therefore omitted. For the special case of homogeneous buyers in the base model, we have proven the analogue to the feasibility condition (44) as part of the Proof of Lemma 1, see the remark following Equations (11) and (12) on Page 28. We have verified the analogues to (46) and (47) in the base model as part of the proof of Lemma 8, see Equation (25) and the subsequent remark on Page 33.

The proof of Proposition 5 utilizes arguments from Lauermann (2012), stated in the following three lemmas. First, proving that the outcomes become competitive is equivalent to proving that limit outcomes are pairwise efficient. This is an immediate extension of the classical characterization result by Shapley and Shubik (1971) for a simple version of the assignment game with transferable utility.

Lemma 12 *Shapley and Shubik (1971). A sequence of feasible outcomes converges to the competitive outcome if it becomes pairwise efficient, that is, if for all v,*

$$\begin{aligned} \liminf_{k\to\infty} EU_k [\theta, v, \beta_k|L] + E\Pi_k \left[\theta', \rho_k|L\right] &\geq v \qquad \textit{for all } \theta, \theta' < 1, \\ \liminf_{k\to\infty} EU_k [\theta, v, \beta_k|H] + E\Pi_k \left[\theta', \rho_k|H\right] &\geq v \qquad \textit{for all } \theta, \theta' > 0. \end{aligned}$$

The following lemma uses standard incentive compatibility implications to characterize payoffs.[38] Note that this characterization applies to traders who share the same belief.

Lemma 13 *For all k, v, v', and θ,*

$$Q_{B,k}^\theta (\theta, v) \left(v - v'\right) \geq V_{B,k} (\theta, v) - V_{B,k} \left(\theta, v'\right) \geq Q_{B,k}^\theta \left(\theta, v'\right) \left(v - v'\right). \tag{48}$$

Proof: Optimality requires that a buyer having valuation v does not have an in-

[38] See, for instance, the characterization of Bayesian incentive compatibility by Mas-Colell, Whinston, and Green (1995), p. 888, Equation 23.D.13.

centive to mimic a buyer having valuation v'. Thus,

$$V_{B,k}(\theta, v) \geq Q^{\theta}_{B,k}(\theta, v')\left(v - P^{\theta}_{B,k}(\theta, v')\right).$$

Interchanging v and v' proves (48). □

Note that Lemma 13 implies that $Q^{\theta}_{B,k}(\theta, v)$ is weakly increasing in v. Payoffs are weakly increasing and Lipschitz-continuous in v, with a slope bounded between zero and one. A useful implication is that $v - \delta_k V_{B,k}(\theta, v)$ is continuous and strictly increasing in v, with a slope between $(1 - \delta_k)$ and one. The expression $v - \delta_k V_{B,k}(\theta, v)$ can be interpreted as a buyer's reservation price or as a buyer's "dynamic type," see Satterthwaite and Shneyerov (2007).

The following Lemma uses the steady-state implications and the no-introspection condition to provide a characterization of the distribution of types in the stock.

Lemma 14 *Take any sequence of sets $\{Z_k\}_{k=1}^{\infty}$, with $Z_k \subset [0, 1]$. If*

$$\liminf_{k\to\infty} d^L \int_{[0,1]\times Z_k} \left(1 - Q^L_{B,k}(\theta, v)\right) dG^L_B > 0 \tag{49}$$

then $\liminf_{k\to\infty} D^L_k \Gamma^L_{B,k}([0, 1] \times Z_k)/S^L_k > 0$. In addition, there is some $\hat{\theta} < 1$ such that $\liminf_{k\to\infty} \Gamma^L_{B,k}([0, \hat{\theta}] \times Z_k)/\Gamma^L_{B,k}([0, 1] \times Z_k) > 0$. If, in addition to (49),

$$\lim_{k\to\infty} d^H \int_{[0,1]\times Z_k} \left(1 - Q^H_{B,k}(\theta, v)\right) dG^H_B = 0, \tag{50}$$

then $\lim_{k\to\infty} \Gamma^L_{B,k}([0, \hat{\theta}] \times Z_k)/\Gamma^L_{B,k}([0, 1] \times Z_k) = 1$ for all $\hat{\theta} \in (0, 1)$.

Note that the lemma holds analogously with H and L interchanged. In that case, $[0, \hat{\theta}]$ is replaced by $[\hat{\theta}, 1]$ in the last two implications.

The first part of the lemma extends an observation from Lauermann (2012, Claim 1): If Z_k is a sequence of non-trivial sets of buyers' valuations such that buyers with such valuations do not trade with probability one in the limit, then the share of these sets in the stock does not vanish. Intuitively, traders who do not trade with probability one in the limit stay in the stock for a long time, and, therefore, these traders make up a positive share of the total stock. The second part of the lemma establishes bounds on the beliefs of these buyers. This part uses the no-introspection condition to capture implications of Bayesian updating. Specifically, if buyers from the set Z_k do not trade with probability one in the low state, then it cannot be the case that most of the buyers in the stock who have such valuations become convinced that the state is high even though it is not. Conversely, if buyers with valuations from the set Z_k would have been able to trade with probability one in the high state but not in the low state, then most of the buyers with such valuations in the stock must be certain that the state is low if the state is indeed low.

Proof: Adding (46) and (47) provides a bound on the mass of buyers,

$$D_k^w \Gamma_{B,k}^w([0,1] \times Z_k)\,(1-\delta_k) \leq d^w G_B^w([0,1] \times Z_k), \tag{51}$$

where the left side corresponds roughly to the number of buyers with values $v \in Z_k$ who die each period and the right side corresponds to the number of such buyers who enter the market. A similar bound is implied for sellers,

$$S_k^w\,(1-\delta_k) \leq 1. \tag{52}$$

Now, suppose that (along some convergent subsequence),

$$\lim_{k\to\infty} d^L \int_{[0,1]\times Z_k} \left(1 - Q_{B,k}^L(\theta, v)\right) dG_B^L = K > 0.$$

Then, (46) and (47) imply that[39]

$$\liminf_{k\to\infty} D_k^L \Gamma_{B,k}^L([0,1] \times Z_k)\,(1-\delta_k) = K. \tag{53}$$

Now, (52) and (53) imply $\liminf_{k\to\infty} D_k^L \Gamma_{B,k}^L([0,1] \times Z_k)/S_k^L \geq K > 0$. The first claim of the lemma follows.

For the second claim, pick any $\hat{\theta} \in (0,1)$ and suppose that (along some convergent subsequence),

$$\lim_{k\to\infty} \frac{\Gamma_{B,k}^L([0,\hat{\theta}] \times Z_k)}{\Gamma_{B,k}^L([0,1] \times Z_k)} = R_{\hat{\theta}}.$$

Thus, combined with (53),

$$\liminf_{k\to\infty} D_k^L \Gamma_{B,k}^L([\hat{\theta},1] \times Z_k)\,(1-\delta_k) = \left(1 - R_{\hat{\theta}}\right) K.$$

The second part of the Lemma holds if $\left(1 - R_{\hat{\theta}}\right) K = 0$. So, suppose that $\left(1 - R_{\hat{\theta}}\right) K > 0$. From no-introspection,

$$\frac{D_k^H \Gamma_{B,k}^H([\hat{\theta},1] \times Z_k)}{D_k^L \Gamma_{B,k}^L([\hat{\theta},1] \times Z_k)} \geq \frac{\hat{\theta}}{1-\hat{\theta}}.$$

[39] Equation (46) implies that $D_k^w \int_{[0,1]\times Z_k} q_{B,k}^w(\theta, v)\, d\Gamma_{B,k}^w$ is bounded, and, hence, $\lim\,(1-\delta_k)\, D_k^w \int_{[0,1]\times Z_k} q_{B,k}^w(\theta, v)\, d\Gamma_{B,k}^w = 0$. Now, (53) follows from (47).

Hence,

$$\liminf_{k\to\infty} D_k^H \Gamma_{B,k}^H([\hat{\theta},1]\times Z_k)\,(1-\delta_k) \geq \frac{\hat{\theta}}{1-\hat{\theta}}\left(1-R_{\hat{\theta}}\right)K. \tag{54}$$

Choose $\hat{\theta}$ such that $\frac{\hat{\theta}}{1-\hat{\theta}}K > d^L G_B^L([0,\hat{\theta}]\times Z_k)$. Then, (51) implies $R_{\hat{\theta}} > 0$. Thus, the second part of the Lemma holds. Finally, suppose that (50) holds. Then, (46) and (47) require that

$$\lim_{k\to\infty} D_k^H \Gamma_{B,k}^H([\hat{\theta},1]\times Z_k)\,(1-\delta_k) = 0.$$

Thus, (54) requires that for all $\hat{\theta} > 0$, $R_{\hat{\theta}} = 1$, as claimed. □

9.3 Proof of Proposition 5

To simplify the exposition, we will consider a subsequence for which some key objects converge pointwise. Specifically, Lemma 13 implies that $V_{B,k}(\theta,v)$ is monotone in v for $\theta \in \{0,1\}$. Thus, there exists a subsequence such that $V_{B,k}(0,v)$ and $V_{B,k}(1,v)$ converge pointwise everywhere by Helly's selection theorem. There exists a further subsequence such that the numbers $\rho_k(0)$, $\rho_k(1)$, $V_{S,k}(0)$, $V_{S,k}(1)$, and $P_{S,k}^H(1)$ converge as well. We show that the limits are independent of the choice of the subsequence. Hence, the sequence itself converges. We identify the convergent subsequence with itself for notational convenience. We indicate the limits using bars over the variables, for instance, $\lim_{k\to\infty} V_{B,k}(0,v) = \bar{V}_B(0,v)$ and, similarly, $\bar{V}_B(1,v)$, $\bar{\rho}(0)$, $\bar{\rho}(1)$, $\bar{P}_S^H(1)$, $\bar{V}_S(0)$, and $\bar{V}_S(1)$.

The proof proceeds through a sequence of claims, which are combined to prove the proposition at the end.

The first claim states conditions under which traders essentially learn the true state, as reflected by the linearity of limit payoffs in beliefs.

Claim 1 *If $\bar{V}_B(0,v') = v' - \bar{\rho}(0)$ for some $v' > \bar{\rho}(0)$, then $\bar{V}_B(0,v) = v - \bar{\rho}(0)$ for all $v > \bar{\rho}(0)$ and*

$$\lim_{k\to\infty} V_{B,k}(\theta,v) = (1-\theta)\,\bar{V}_B(0,v) + \theta\bar{V}_B(1,v) \qquad \forall(\theta,v). \tag{55}$$

If $\bar{V}_S(1) = \bar{P}_S^H(1)$ and $\bar{V}_S(0) = \bar{\rho}(0)$, then

$$\lim_{k\to\infty} V_{S,k}(\theta) = (1-\theta)\,\bar{V}_S(0) + \theta\bar{V}_S(1) \qquad \forall\theta. \tag{56}$$

Proof: We consider the buyers' side first. By the monotonicity of ρ_k, $\lim_{k\to\infty} V_{B,k}(\theta,v) = 0$ for all $v \leq \bar{\rho}(0)$. We therefore consider $v > \bar{\rho}(0)$.

The hypothesis implies $\bar{\rho}(0) < 1$ and $\bar{V}_B(0, v') > 0$. Therefore, $\bar{V}_B(0, v') = v' - \bar{\rho}(0)$ and $P^L_{B,k}(0, v) \geq \rho_k(0)$ together requires $\lim_{k\to\infty} Q^L_{B,k}(0, v) = 1$.

The claim holds for $\theta = 1$ by definition. For $\theta = 0$, $\lim_{k\to\infty} Q^L_{B,k}(0, v') = 1$ and Lemma 13 together imply that $\bar{V}_B(0, v) = v - \bar{\rho}(0)$ for all $v > \bar{\rho}(0)$, as claimed.

We now consider $\theta \in (0, 1)$. Take any $v > \bar{\rho}(0)$. From the convexity of value functions in beliefs (see the proof of Lemma 5),

$$V_{B,k}(\theta, v) \leq (1-\theta) V_{B,k}(0, v) + \theta V_{B,k}(1, v).$$

We show that the equation holds with equality in the limit, observing that the payoffs from the following strategy provide a lower bound on $V_{B,k}(\theta, v)$: Given some $\varepsilon > 0$ and k large enough, bid $\rho_k(0) + \varepsilon$ for T_k periods, where $(\delta_k)^{T_k} \geq 1 - \varepsilon \geq (\delta_k)^{T_k+1}$. Then, bid according to $\beta_k(1, v)$ forever after. We consider the expected payoffs in the low state in the limit. From $\lim_{k\to\infty} Q^L_{B,k}(0, v) = 1$, $\lim_{k\to\infty} \rho_k(0) = \bar{\rho}(0)$, and $\bar{V}_B(0, v) = v - \bar{\rho}(0)$, the buyer's payoff from this strategy is at least $(1-\varepsilon)(v - \bar{\rho}(0) - \varepsilon)$ in the limit. Now, we consider the high state. With probability $(1-\varepsilon)$ the buyer either trades within the first T_k periods for a payoff of at least $v - \bar{\rho}(0) - \varepsilon$ in the limit or the buyer stays in the market for T_k periods and then expects a payoff of $\bar{V}_B(1, v)$. The monotonicity of ρ_k implies $\bar{V}_B(1, v) \leq v - \bar{\rho}(0)$. Therefore, the payoff from the described strategy in the limit is at least $(1-\varepsilon)\left(\bar{V}_B(1, v) - \varepsilon\right)$. Since ε is arbitrary, the claim follows.

Similar reasoning applies to the sellers. If $\bar{P}^H_S(1) = 0$, then the claim is immediate from $P^H_{S,k}(1) \geq \rho_k(1) \geq \rho_k(0)$. So, suppose $\bar{P}^H_S(1) > 0$. From $\bar{V}_S(1) = \bar{P}^H_S(1)$, it must be that $\lim_{k\to\infty} Q^H_{S,k}(1) = 1$. Consider the profit from the following strategy. Given $\varepsilon > 0$, set $\rho_k(1)$ for T^k periods, followed by setting $\rho_k(0)$ forever after, with T^k defined as before. In the high state, the strategy ensures the seller a payoff of $\bar{V}_S(1)$ in the limit. In the low state, with probability $(1-\varepsilon)$, the seller either trades within the first T^k periods at a price at least $\rho_k(1)$ or stays in the market for all periods and then expects $V_{S,k}(0)$. Thus, the payoff is at least $\bar{\rho}(0) - 2\varepsilon$ in the low state, proving the claim. $\square$

If the hypothesis of Claim 1 and, thus, (55) are true, then for all v,

$$\lim_{k\to\infty} EU_k[\theta, v, \beta_k | L] = \bar{V}_B(0, v) = \max\{0, v - \bar{\rho}(0)\} \qquad \forall\, \theta < 1, \tag{57}$$

$$\lim_{k\to\infty} EU_k[\theta, v, \beta_k | H] = \bar{V}_B(1, v) \qquad \forall\, \theta > 0. \tag{58}$$

This follows immediately from the combination of (55) and the optimality implications $EU_k[\theta, v, \beta_k | L] \leq V_{B,k}(0, v)$ and $EU_k[\theta, v, \beta_k | H] \leq V_{B,k}(1, v)$.

Claim 2 $\bar{V}_S(0) = \bar{\rho}(0)$.

Proof: We first show $\bar{V}_S(0) \leq \bar{\rho}(0)$ by observing that

$$\delta_k E\Pi_k[0, \rho_k(0)|L] \leq \rho_k(0) \qquad \forall k.$$

By contradiction. Suppose $\delta_k E\Pi_k[0, \rho_k(0)|L] > \rho_k(0)$. We show that it is a profitable deviation to set $r_k = \delta_k E\Pi_k[0, \rho_k(0)|L]$ instead of $\rho_k(0)$.

Step 1: When setting $\rho_k(0)$, there is a strictly positive chance to trade at a price strictly below $\delta_k E\Pi_k[0, \rho_k(0)|L]$.

Proof: The monotonicity of ρ_k and the sequential rationality of bids imply $\delta_k E\Pi_k[0, \rho_k(0)|L] > \beta_k(\theta, v, \rho_k(0)) \geq \rho_k(0)$ for all v for which $\rho_k(0) < v < \delta_k E\Pi_k[0, \rho_k(0)|L]$.

Step 2: If $\beta_k\left(\theta_k^I(\rho_k(0), \theta), v, \rho_k(0)\right) \geq r_k$ for some type (θ, v), then $\beta_k\left(\theta_k^I(r_k, \theta), v, r_k\right) \geq r_k$.

Proof: First, note that for all $r_k > \rho_k(0)$, the interim belief $\theta_k^I(r_k, \theta) \geq \theta_k^I(\rho_k(0), \theta)$ for all θ. This follows from the monotonicity of ρ_k if $r_k = \rho_k(\theta)$ for some θ. If r_k is off the equilibrium path, then $\theta_k^I(r_k, \theta) = \theta$ while $\theta_k^I(\rho_k(0), \theta) \leq \theta$. Thus, if $\beta_k\left(\theta_k^I(\rho_k(0), \theta), v, \rho_k(0)\right) \geq r_k$ for some type (θ, v), then the monotonicity of β_k in interim believe and in the reserve price implies $\beta_k\left(\theta_k^I(r_k, \theta), v, r_k\right) \geq r_k$.

Step 1 and 2 imply that setting $r_k = \delta_k E\Pi_k[0, \rho_k(0)|L]$ strictly dominates setting $\rho_k(0)$. This is because either (i) there is trade under $\rho_k(0)$ at a price strictly below $\delta_k E\Pi_k[0, \rho_k(0)|L]$ (so that it would be strictly better to continue), or, (ii), there is trade under $\rho_k(0)$ at a price weakly above $\delta_k E\Pi_k[0, \rho_k(0)|L]$ (and then the price-setting highest bid would be weakly higher under r_k by Step 2), or, (iii), there is no trade under $\rho_k(0)$ (and then there is either no trade under r_k as well, or there is trade at a price higher than $\delta_k E\Pi_k[0, \rho_k(0)|L]$.) By Step 1, there is a strictly positive probability of case (i), which implies that bidding r_k yields indeed strictly higher payoffs.

We now show $\bar{V}_S(0) \geq \bar{\rho}(0)$. Given any $\varepsilon > 0$, let $r_k^\varepsilon = \rho_k(0) - \varepsilon$. When setting r_k^ε, all buyers with values $v \geq r_k^\varepsilon$ bid $b \geq r_k^\varepsilon$ by the sequential rationality of bids and by $EU_k[\theta, v, \beta_k|\theta] \leq v - r_k^\varepsilon$ from $\rho_k(\theta) > r_k^\varepsilon$ for all θ. Let $A_k = \{(\theta, v) : v \in (r_k^\varepsilon, \rho_k(0))\}$. By the definition of $\rho_k(0)$ and the individual rationality of bids, $\left(1 - Q_{B,k}^L(\theta, v)\right) = 1$ for all $(\theta, v) \in A_k$. From Lemma 14, $\liminf_{k\to\infty} D_k^L \Gamma_{B,k}^L(A_k)/S_k^L > 0$. Hence, when setting r_k^ε, the per-period trading probability $\liminf_{k\to\infty} q_{S,k}[r_k^\varepsilon|L] > 0$, and, so, $\lim_{k\to\infty} Q_{S,k}[r_k^\varepsilon|L] = 1$. Thus, the limit payoff from setting r_k^ε is at least $\rho_k(0) - \varepsilon$. From ε being chosen arbitrarily, it follows that equilibrium payoffs are at least $\rho_k(0)$, as claimed. □

Claim 3 $\lim_{k\to\infty} \int_{[0,1]\times[\bar{\rho}(0),1]} \left(1 - Q_{B,k}^L(\theta, v)\right) dG_B^L = 0.$

Proof: By contradiction. Suppose $\int_{[0,1]\times[\bar{\rho}(0),1]}\left(1-Q^L_{B,k}(\theta,v)\right)dG^L_B>0$ along some converging subsequence (without loss of generality, the sequence itself).

Step1. There are v',v'', $\bar{\rho}(0)<v'<v''$ and some sequence $\{Z_k\}_{k=1}^{\infty}$, $Z_k\subset[v',v'']$, such that $\liminf_{k\to\infty}F^L(Z_k)>0$ and for some $\bar{Q}<1$ and K, $Q^L_{B,k}(\theta,v)\leq\bar{Q}$ for all $v\in Z_k$ and $k\geq K$.

Proof: This follows from the hypothesis.

Step 2. There is some $\varepsilon_1>0$ such that $\bar{V}_B(0,v)<v-\bar{\rho}(0)-\varepsilon_1$ for all $v\in[v',v'']$.

Proof: This follows from Claim 1 and Lemma 13. Otherwise, if $\bar{V}_B(0,v')=v'-\bar{\rho}(0)$ for some $v'>\bar{\rho}(0)$, then Lemma 13 implies $\bar{V}_B(0,v)=\max\{0,v-\bar{\rho}(0)\}$. Hence, Claim 1 requires $\lim_{k\to\infty}EU_k[\theta,v,\beta_k|L]=\max\{0,v-\bar{\rho}(0)\}$, from where $\lim_{k\to\infty}Q^L_{B,k}(\theta,v)=1$ for all $v>\bar{\rho}(0)$, which is a contradiction to the starting hypothesis.

Case 1: Suppose $\bar{\rho}(0)<\bar{\rho}(1)$.

Pick ε_2 such that $0<\varepsilon_2<\min\{\varepsilon_1,\bar{\rho}(1)-\bar{\rho}(0)\}$.

Step 3 (Case 1). $\lim_{k\to\infty}\sup EU[\theta,v,\beta_k|\theta]<v-\bar{\rho}(0)-\varepsilon_2$ for all $v\in[v',v'']$ and for all θ.

Proof: This follows from $EU[\theta,v,\beta_k|\theta]\leq(1-\theta)V_{B,k}(0,v)+\theta V_{B,k}(1,v)$, $\lim_{k\to\infty}V_{B,k}(1,v)\leq v-\bar{\rho}(1)$, Step 2, and the definition of ε_2.

Step 4 (Case 1). Let $r^{\varepsilon}_k=\rho_k(0)+\varepsilon_2$. Then, the per-period trading probability $\liminf_{k\to\infty}q^L_{S,k}(r^{\varepsilon}_k)>0$.

Proof: First, from Step 3 and from the sequential rationality of bids, $\beta_k\left(\theta^I_r(\theta,r^{\varepsilon}_k),v,r^{\varepsilon}_k\right)\geq r^{\varepsilon}_k$ for all θ and all $v\in[v',v'']$. Thus,

$$q^L_{S,k}(r^{\varepsilon}_k)\geq 1-e^{D^L_k\Gamma^L_{B,k}([0,1]\times[v',v''])/S^L_k}.$$

Second, from Lemma 14,

$$\liminf_{k\to\infty}D^L_k\Gamma^L_{B,k}\left([0,1]\times\left[v',v''\right]\right)/S^L_k>0.$$

Hence, $\liminf_{k\to\infty}q^L_{S,k}(r^{\varepsilon}_k)>0$, as claimed.

Step 4 implies that $\lim_{k\to\infty}Q^L_{S,k}(r^{\varepsilon}_k)=1$. Thus, the limit payoff from offering r^{ε}_k is larger than $\bar{\rho}(0)+\varepsilon_2$ and so must be the equilibrium payoff. This is in contradiction to Claim 2.

Case 2: Suppose $\bar{\rho}(0)=\bar{\rho}(1)$.

Pick some $\hat{\theta}$ such that the second part from Lemma 14 holds.

Step 5 (Case 2). There is some $\varepsilon'>0$ such that $\lim_{k\to\infty}\sup EU_k[\theta,v,\beta_k|\theta]<v-(\rho_k(0)+\varepsilon')$ for all $\theta<\hat{\theta}$.

Proof: This follows from $\lim_{k\to\infty}\sup EU_k[\theta, v, \beta_k|L] < v - \bar{\rho}(0) - \varepsilon_1$ (Step 2) and $\lim_{k\to\infty}\sup EU_k[\theta, v, \beta_k|H] \leq v - \bar{\rho}(0)$ (by the monotonicity of ρ_k).[40]

Step 6 (Case 2). Let $r_k^{\varepsilon'} = \rho_k(0) + \varepsilon'$. Then, $\liminf_{k\to\infty} q_{S,k}^L(r_k^{\varepsilon'}) > 0$.

Proof: Suppose not. First, sufficiently deep into the sequence, $r_k^{\varepsilon'} > \rho_k(1)$. Hence, passive beliefs implies $\theta^I(\theta, r_k^{\varepsilon'}) = \theta$ for all θ. Second, $\liminf_{k\to\infty} q_{S,k}^L(r_k^{\varepsilon'}) = 0$ implies that the probability that a buyer faces a competing bid $b \geq r_k^{\varepsilon'}$ converges to zero. Hence, sufficiently deep into the sequence, bidding $b = r_k^{\varepsilon'}$ dominates not participating because $\liminf_{k\to\infty} v - r_k^{\varepsilon'} > \lim_{k\to\infty}\sup EU[\theta, v, \beta_k|\theta]$ by Step 5. Therefore, sequential rationality requires $\beta_k(\theta^I(\theta, r_k^{\varepsilon'}), v, r_k^{\varepsilon'}) \geq r_k^{\varepsilon'}$ for all $v \in [v', v'']$ and $\theta < \hat{\theta}$. Now, $\liminf_{k\to\infty} q_{S,k}^L(r_k^{\varepsilon'}) > 0$ follows from the choice of $\hat{\theta}$ and Lemma 14, analogously to Step 4.

Step 6 implies that the limit payoff from offering r_k^{ε} is larger than $\bar{\rho}(0) + \varepsilon'$ yielding again a contradiction to Claim 2. Thus, Claim 3 must be true. $\square$

Claim 4 *Equations (57) and (58) hold. In addition,* $\bar{\rho}(0) > 0$.

Proof: Claim 3 and Lemma 13 together imply that $\bar{V}_B(0, v) = v - \bar{\rho}(0)$ for all $v > \bar{\rho}(0)$. Claim 1 and the subsequent remarks imply that (57) and (58) hold.

Furthermore, from Claim 3, the mass of buyers who enter in the low state and trade is equal to $1 - F^L(\bar{\rho}(0))$ in the limit. Thus, feasibility of the steady state requires that a mass $1 - F^L(\bar{\rho}(0))$ of sellers enters and trade; see (44). Since the mass of sellers is one, this requires $1 - F^L(\bar{\rho}(0)) \leq 1$. By definition of p_*^w and the full support assumption on F^L, $1 - F^L(\bar{\rho}(0)) \leq 1$ requires $\bar{\rho}(0) \geq p_*^w$. The claim follows from $p_*^w > 0$. $\square$

Claim 5 $\lim_{k\to\infty} EU_k[\theta, v, \beta_k|H] \geq v - \bar{P}_S^H(1)$ *for all* $\theta > 0$ *and* v.

Proof: Fix any $\varepsilon > 0$.

Step 1. For every reserve price r, there must be a positive probability in the limit that there is either no bid or that the highest bid is below $\bar{P}_S^H(1) + \varepsilon$.

Proof: From optimality of $\rho_k(1)$ and from $\bar{V}_S(1) \leq \bar{P}_S^H(1)$, it follows that there is no r_k such that when setting reserve price r_k, with probability converging to one the second highest bid is above $\bar{P}_S^H(1) + \varepsilon$. This and the Poisson distribution of bids implies Step 1.

Step 2. $\bar{V}_B(1, v) \geq v - \bar{P}_S^H(1) - \varepsilon$ for all v.

[40] Note that $\varepsilon' < \varepsilon_1$ may hold if $\bar{V}_B(0, v) < \bar{V}_B(1, v)$.

Proof: The strategy of always bidding $b = \bar{P}_S^H(1) + \varepsilon$ ensures that the bidder will eventually win an auction and pay at most $\bar{P}_S^H(1) + \varepsilon$. This is because $\rho_k(\theta) \leq \bar{P}_S^H(1)$ for all θ (by the monotonicity of ρ_k), and, by Step 1, with positive probability, there is either no competing bid at all or the highest competing bid is below $\bar{P}_S^H(1)+\varepsilon$.

Because ε is arbitrary, Step 2 implies $\bar{V}_B(1, v) \geq v - \bar{P}_S^H(1)$. Now the claim follows from implication (58) of Claim 1 . □

Claim 6 $\bar{V}_S(1) = \bar{P}_S^H(1)$

Proof: Step 1 $\bar{V}_S(1) \geq \bar{\rho}(0)$

Proof: Analogously to the last part of the proof of Claim 2, the seller could lower its price to $\rho_k(0) - \varepsilon$ and trade with probability one in the limit for all $\varepsilon > 0$. Hence, $\bar{V}_S(1) \geq \bar{\rho}(0)$.

Step 1 implies the claim if $\bar{P}_S^H(1) = \bar{\rho}(0)$. So, suppose $\bar{P}_S^H(1) > \bar{\rho}(0)$ in the following. We prove the claim by contradiction and assume $\bar{\rho}(0) \leq \bar{V}_S(1) < \bar{P}_S^H(1)$.

Step 2. $\bar{V}_S(1) < \bar{P}_S^H(1)$ implies $\lim_{k\to\infty} \sup Q_{S,k}^H(1) < 1$ and $\bar{P}_S^H(1) = \bar{\rho}(1)$

Proof: Throughout the proof, we abbreviate $\bar{P}_S^H(1) = \bar{P}_S^H$.

(i) $\lim_{k\to\infty} \sup Q_{S,k}^H(1) < 1$ follows from $\bar{P}_S^H > 0$.

(ii) $\lim_{k\to\infty} \sup Q_{S,k}^H(1) < 1$ implies that $\lim_{k\to\infty} q_{S,k}^H(1) = 0$, that is, the probability that at least one buyer bids above $\rho_k(1)$ converges to zero. The Poisson-distribution implies that the probability that two buyers bid above $\rho_k(1)$ is infinitely smaller than the probability that a single buyer bids above $\rho_k(1)$ in the limit. Therefore, $\bar{P}_S^H = \bar{\rho}(1)$, as claimed.

Pick any p' such that

$$\bar{V}_S(1) < p' < \bar{P}_S^H.$$

In the following, we show that setting p' is a profitable deviation. We are done if $\lim_{k\to\infty} \sup q_{S,k}^H(p') > 0$. So, suppose that

$$\lim_{k\to\infty} q_{S,k}^H(p') = 0. \tag{59}$$

Step 3. There are $v'' > v' > \bar{\rho}(0)$ such that $\lim_{k\to\infty} EU_k[\theta, v, \beta_k|\theta] > v - \bar{P}_S^H$ for all $v \in [v', v'']$ and all θ. In addition, there is a $\hat{\theta} < 1$ such that $v - p' > \lim_{k\to\infty} EU_k[\theta, v, \beta_k|\theta]$ for all $v \in [v', v'']$ and $\hat{\theta} < 1$.

Proof: (i) $\bar{V}_B(1,1) \leq 1 - \bar{P}_S^H$. Otherwise, $\bar{V}_B(1,1) > 1 - \bar{P}_S^H$ and Lemma 13 imply $\bar{V}_B(1, v) > v - \bar{P}_S^H$ for all v. Therefore, Claim 1 implies $\lim_{k\to\infty} EU[\theta, v, \beta_k|\theta] > v - \bar{P}_S^H$ for all v and all θ, from where $\beta_k(\theta, v) < \rho_k(1)$for all k large enough implies $\bar{V}_S(1) = 0$. But $\bar{V}(1) \geq \bar{\rho}(0) > 0$ by Step 1 and Claim 4.

(ii) Pick p_l and p_h such that $\bar{P}_S^H > p_h > p_l > p'$. By $\bar{V}_B(1,1) > 1 - \bar{P}_S^H$, $\bar{V}_B(1,0) = 0$, and the continuity of payoffs implied by Lemma 13, the intermediate value theorem implies that there are v_k' and v_k'' such that $\delta_k EU[1, v_k', \beta_k|H] = v_k' - p_l$ and $\delta_k EU[1, v_k'', \beta_k|H] = v_k'' - p_h$ for all k large enough. Let $v' = \lim_{k\to\infty} v_k'$ and $v'' = \lim_{k\to\infty} v_k''$ be the limits. By Lemma 13 and the following remark, the limits $v'' > v'$. By Lemma 13, $v - p_h \leq \lim_{k\to\infty} EU_k[1, v, \beta_k|H]$ for all $v \in [v', v'']$. From $\bar{P}_S^H > \bar{\rho}(0)$ and Claim 4, $\lim_{k\to\infty} EU_k[\theta, v, \beta_k|\theta] \geq (1-\theta)(v - \bar{\rho}(0)) + \theta(v - p_h) > v - \bar{P}_S^H$ for all θ, proving the first part.

In addition, by Lemma 13, $v - p_l \geq \lim_{k\to\infty} EU_k[1, v, \beta_k|H]$ for all $v \in [v', v'']$. From Claim 4 $\lim_{k\to\infty} EU_k[\theta, v, \beta_k|\theta] \leq (1-\theta)(v - p') + \theta(v - p_h)$. Thus, $p_l > p'$ implies that second part holds by choosing $\hat{\theta}$ sufficiently close to one.

Step 4. There is some $\hat{Q} < 1$ such that $\lim_{k\to\infty} Q_{B,k}^H(\theta, v, \beta_k) \leq \hat{Q}$ for all $v \in [v', v'']$.

Proof: Suppose not. Then there is some sequence $v_k \in [v', v'']$ and θ_k such that $\lim_{k\to\infty} Q_{B,k}^H(\theta_k, v_k) = 1$. Let $\bar{v} = \lim_{k\to\infty} v_k$. From Step 3 and Claim 4 (and continuity of limit payoffs), $\lim_{k\to\infty} EU_k[\theta_k, v_k, \beta_k|H] = \lim_{k\to\infty} \bar{V}_B(1, \bar{v}) > \bar{v} - \bar{P}_S^H$. Hence, $\lim_{k\to\infty} P_{B,k}^H(\theta_k, v_k) < \bar{P}_S^H$. Thus, if $v = 1$ uses the same sequence of bids as (θ_k, v_k), the buyer trades with probability one at a price strictly below $\bar{P}_S^H$. Hence, equilibrium payoffs must be bounded below, $\bar{V}_B(1,1) > 1 - \bar{P}_S^H$. This is in contradiction to the upper bound derived in proof of Step 3, part (i).

Step 5. $\liminf_{k\to\infty} D_k^H \Gamma_{B,k}^H([\theta, 1] \times [v', v'']) / S_k^H > 0$ for all $\theta < 1$.

Proof: From Claim 3 and Step 4, buyers with values $v \in [v', v'']$ trade with probability one in the low state, but they trade with probability less than one in the high state. The step now follows from Lemma 14.

Step 6. $\lim_{k\to\infty} \theta_k^I(\theta, p') = 0$ for all $\theta < 1$.

Proof: Suppose, by way of contradiction, that $\lim_{k\to\infty} \theta_k^I(\theta', p') > 0$ for some $\theta' < 1$.

(i) There is some $\theta'' < 1$ such that $\lim_{k\to\infty} \theta_k^I(\theta, p') > \hat{\theta}$ for all $\theta \geq \theta''$, with $\hat{\theta}$ chosen as in Step 3. This follows from the definition of interim beliefs.

(ii) For k sufficiently large, all buyers with types $(\theta, v) \in [\theta'', 1] \times [v', v'']$ bid $\beta_k(\theta, v, p') \geq p'$. This is because not participating is dominated by bidding $b = 1$: The payoff from bidding $b = 1$ converges to $v - p'$, by (59). The continuation payoff from not participating is $\delta_k EU[\theta_k^I(\theta, p'), v|\theta_k^I(\theta, p')]$. From Step 3 and choice of θ'', the continuation payoff $\lim_{k\to\infty} \sup \delta_k EU[\theta_k^I(\theta, p'), v|\theta_k^I(\theta, p')] < v - p'$ for all $\theta \geq \theta''$.

(iii) Combining (ii) and Step 5 implies $\liminf_{k\to\infty} q_{S,k}^H(p') > 0$, a contradiction to (59).

Step 7. There is some sequence $\{\theta_k\}$ such that $\rho_k(\theta_k) = p'$ for all k large enough and $\lim_{k\to\infty}\theta_k = 0$.

Proof: (i) If there is no sequence $\{\theta_k\}$ such that $\rho_k(\theta_k) = p'$ for all k large enough, then the assumption that beliefs are passive implies that $\theta_k^I(p', \theta) = \theta$, in contradiction to Step 6.

(ii) Suppose $\lim_{k\to\infty}\theta_k \neq 0$. Then there is some $\theta' > 0$ such that $\theta_k \geq \theta'$ for all k sufficiently large (along some subsequence). From the definition of interim beliefs (43),

$$\theta_k^I(p', \theta) \geq \frac{\theta}{1-\theta}\frac{S_k^H\left(1 - \Gamma_{S,k}^H(\theta')\right)}{S_k^L\left(1 - \Gamma_{S,k}^L(\theta')\right)} \geq \frac{\theta}{1-\theta}\frac{\theta'}{1-\theta'} > 0,$$

where the inequalities follow from the no-introspection condition. This is in contradiction to Step 6.

Remark: The proof of Step 7 is the critical application of passive beliefs.

Step 8. $\lim_{k\to\infty} Q_{k,S}^L(p') > 0$.

Proof: The payoff to a seller with belief θ_k from offering $\rho_k(\theta_k) = p'$ is

$$(1-\theta_k)\left(q_k^L p_k^L + \left(1 - q_k^L\right)\delta_k E\Pi_k\left(\theta_S^+, \rho_k|L\right)\right) + \theta_k\left(q_k^H p_k^H + \left(1 - q_k^H\right)\delta_k E\Pi_k\left(\theta_S^+, \rho_k|H\right)\right),$$

where p_k^w is the expected price conditional on trading when setting p', q_k^w is the (per-period) trading probability, and $\theta_S^+ = \theta_{S,k}^+(\theta_k, p')$ is the updated belief. Since $\rho_k(\theta_k) = p'$, it must not be optimal to 'skip' offering p'. Skipping p' and setting $\rho_k\left(\theta_S^+\right)$ immediately (and then continuing according to the equilibrium sequence of bids) yields payoffs

$$(1-\theta_k) E\Pi_k\left(\theta_S^+, \rho_k|L\right) + \theta_k E\Pi_k\left(\theta_S^+, \rho_k|H\right).$$

Comparing payoffs, rearranging terms, and dividing by $1-\delta_k$ shows that optimality of p' requires

$$(1-\theta_k)\frac{q_k^L}{1-\delta_k}p_k^L + \theta_k\frac{q_k^H}{1-\delta_k}p_k^H \geq (1-\theta_k) E\Pi_k\left(\theta_S^+|L\right) + \theta_k E\Pi_k\left(\theta_S^+|H\right). \quad (60)$$

From $\lim_{k\to\infty}\theta_k = 0$, it must be that $\liminf_{k\to\infty} E\Pi_k\left(\theta_S^+, \rho_k|L\right) = V_S(0)$.[41] Using Claims 2 and 4, $V_S(1) \geq \bar{\rho}(1) > 0$. Thus, the limit of the right side (60) must be strictly positive. Hence, the limit of the left side must be strictly positive, too. This implies that either

$$\liminf_{k\to\infty}\frac{q_k^L}{1-\delta_k}p_k^L > 0, \quad (61)$$

[41] From $\lim q_{S,k}^w(p') = 0$, and $\lim\theta_k = 0$, $\lim\theta_{S,k}^+(\theta_k, p') = 0$.

and/or

$$\lim_{k\to\infty}\frac{q_k^H}{1-\delta_k}=\infty. \tag{62}$$

Suppose (61). Then, the claim follows from

$$\liminf_{k\to\infty} Q_{k,S}^L\left(p'\right)=\liminf_{k\to\infty}\frac{\frac{q_k^L}{1-\delta_k}}{\frac{q_k^L}{1-\delta_k}+1}>0. \tag{63}$$

Suppose (62). Then, $\lim_{k\to\infty} Q_{k,S}^H\left(p'\right)=1$, analogous to (63). Hence, $\bar{V}_S\left(1\right)\geq p'$, in contradiction to the choice of p'.

We now prove the final step, which implies that setting p' is a profitable deviation. Thus, this step establishes the claim.

Step 9. $\lim_{k\to\infty} Q_{k,S}^H\left(p'\right)=1$.

Proof: Let $A_k=\left\{\left(\theta,v\right)|\beta_k\left(\theta,v\right)\geq p'\right\}$. From $p'>\bar{\rho}\left(0\right)$, for all $\left(\theta_k,v_k\right)\in A_k$, the interim belief must be such that $\liminf_{k\to\infty}\theta_k^I\left(\theta_k,p'\right)>0$. Combined with Step 6, this requires $\lim_{k\to\infty}\theta_k=1$. No introspection requires

$$\frac{D_k^H\left(\Gamma_{B,k}^H\left(A_k\right)\right)}{D_k^L\left(\Gamma_{B,k}^L\left(A_k\right)\right)}\geq\inf_{\theta|(\theta,v)\in A_k}\frac{\theta}{1-\theta}.$$

Hence,

$$\lim_{k\to\infty}\frac{D_k^H\left(\Gamma_{B,k}^H\left(A_k\right)\right)}{D_k^L\left(\Gamma_{B,k}^L\left(A_k\right)\right)}=\infty. \tag{64}$$

Case 1: $S_k^H\leq S_k^L$.

From $q_k^w\left(p'\right)=1-e^{-D_k^w\left(\Gamma_{B,k}^w\left(A_k\right)\right)/S_k^w}$, $\lim_{k\to\infty}\frac{S_k^H D_k^H\left(\Gamma_{B,k}^H\left(A_k\right)\right)}{S_k^H D_k^L\left(\Gamma_{B,k}^L\left(A_k\right)\right)}=\infty$ and $\liminf_{k\to\infty}\frac{q_k^L}{1-\delta_k}>0$ together imply $\lim_{k\to\infty}\frac{q_k^H}{1-\delta_k}=\infty$. Thus, $\lim_{k\to\infty} Q_{k,S}^H\left(p'\right)=1$, proving Step 9.

Case 2: $S_k^H>S_k^L$.[42] We prove that

$$\limsup_{k\to\infty} S_k^H/S_k^L<\infty. \tag{65}$$

The bound (65) and (64) imply Step 9.

The main observation is the following:

Step 9b. For every $\theta_*>0$ there is some $\bar{Q}<1$ such that $\lim_{k\to\infty}\sup Q_{S,k}^H\left(\theta\right)<\bar{Q}$ and $\lim_{k\to\infty}\sup Q_{S,k}^L\left(\theta\right)<\bar{Q}$ for all $\theta\geq\theta_*$.

Given θ_*, pick any sequence $\{\theta_k\}$ with $\theta_k\geq\theta_*$ for each element. We show that $\lim_{k\to\infty}\sup Q_{S,k}^H\left(\theta_k\right)<1$ and $\lim_{k\to\infty}\sup Q_{S,k}^L\left(\theta_k\right)<1$, which implies the Step.

[42] We believe that the case $S_k^H>S_k^L$ is impossible in equilibrium but we have been unable to verify this conjecture.

Given θ_k, let $\left\{r_k^t\right\}_{t=1}^{\infty}$ be the sequence of reserve price implied by ρ_k and $\theta_{S,k}^+$, and let $\left\{\theta_k^t\right\}$ be the implied posteriors, for example, $r_k^1 = \rho_k\left(\theta_k\right)$, $r_k^2 = \rho_k\left(\theta_{S,k}^+\left(\theta_k, r_k^1\right)\right)$, and $\theta_k^1 = \theta_k$, $\theta_k^2 = \theta_{S,k}^+\left(\theta_k, r_k^t\right)$. Let T^k be the first period after entry in which a reserve price below p' is offered, that is $r_k^t \geq p'$ for all $t < T^k$ and $r^{T^k} < p'$.

If the seller trades before T^k, the price conditional on trading is larger than p'. Thus, the seller must not be able to trade with probability one before T^k in the limit, that is,

$$\sum_{t=1}^{T^k-1} q_{S,k}^H\left(r_k^t\right) \prod_{i=1}^{i=t-1}\left(1-q_{S,k}^H\left(r_k^i\right)\right)\delta_k \tag{66}$$

does not converge to one. Otherwise, the reserve price strategy $\left\{r_k^t\right\}_{t=1}^{\infty}$ implies payoffs of at least p' in the limit. Hence, $\lim_{k\to\infty} V_k^S(1) \geq p'$, in contradiction to the construction of p'. The probability that the seller dies without trading before reaching period T^k is

$$\sum_{t=1}^{T^k-1}\left(1-\delta_k\right)\left(1-q_{S,k}^H\left(r_k^t\right)\right) \prod_{i=1}^{i=t-1}\left(1-q_{S,k}^H\left(r_k^i\right)\right)\delta_k, \tag{67}$$

and the probability that the seller reaches period T^k, neither trading nor dieing before is

$$\prod_{i=1}^{T^k-1}\left(1-q_{S,k}^H\left(r_k^i\right)\right)\delta_k. \tag{68}$$

Step 7 requires that the belief of the seller in period T^k must converge to zero. Bayes' formula implies that the ratio of the posterior conditional on staying for T^k periods satisfies

$$\frac{\theta_k^{T^k}}{1-\theta_k^{T^k}} = \frac{\theta_k}{1-\theta_k}\frac{\prod_{i=1}^{T^k-1}\left(1-q_{S,k}^H\left(r_k^i\right)\right)\delta_k}{\prod_{i=1}^{T^k-1}\left(1-q_{S,k}^L\left(r_k^i\right)\right)\delta_k}.$$

Thus, $\lim_{k\to\infty}\theta_k^{T^k} = 0$ and $\theta_k \geq \theta_* > 0$ require that $\lim_{k\to\infty}(68) = 0$. Since (66), (67), and (68) add up to one, and $\lim_{k\to\infty}\sup(66) < 1$, it must be that $\liminf_{k\to\infty}(67) > 0$. Now, $\liminf_{k\to\infty}(67) > 0$ implies that the seller trades with probability less than one, $\lim_{k\to\infty}\sup Q_{S,k}^H\left(\theta_k\right) < 1$. This argument also establishes that $\lim_{k\to\infty}\sup Q_{S,k}^L\left(\theta_k\right) < 1$. Thus, Step 9b holds.

We now show (65). First, the steady-state conditions imply the following: If

$$\lim_{k\to\infty}\int_{\theta\in[\theta_*,1]}\left(1-Q_{S,k}^L(\theta)\right)dG_S^L = K > 0$$

then $\liminf_{k\to\infty} S_k^L (1-\delta_k) \geq K$. This follows analogously to Lemma 14. Similarly, in every steady state, $\lim_{k\to\infty} \sup S_k^H (1-\delta_k) \leq 1$. Thus, by Step 10 and the absolute continuity of G_S^w, there is θ_* such that $\liminf_{k\to\infty} S_k^L (1-\delta_k) \geq \left(1-\bar{Q}\right)\left(1-G_S^L(\theta_*)\right) > 0$. Hence, the desired bound (65) follows,

$$\liminf_{k\to\infty} \frac{S_k^H}{S_k^L} = \liminf_{k\to\infty} \frac{S_k^H (1-\delta_k)}{S_k^L (1-\delta_k)} \leq \frac{1}{\left(1-\bar{Q}\right)\left(1-G_S^L(\theta_*)\right)} < \infty.$$

This proves the claim. □

Proof of Proposition 5: Claims 5 and 6, combined with Claims 1 and 2 imply that

$$\liminf_{k\to\infty} EU_k [\theta, v, \beta_k | H] + E\Pi_k \left[\theta', \rho_k | H\right] \geq v - \bar{P}_S^H(1) + \bar{P}_S^H(1) \qquad \text{for all } \theta, \theta' > 0.$$

Likewise, Claims 1, 2, 4, and 6 imply

$$\liminf_{k\to\infty} EU_k [\theta, v, \beta_k | L] + E\Pi_k \left[\theta', \rho_k | L\right] \geq v - \bar{\rho}(0) + \bar{\rho}(0) \qquad \text{for all } \theta, \theta' < 1.$$

Thus, the conditions of Lemma 12 hold and Proposition 5 follows. □

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